MAECO-404

Mathematics

MA ECONOMICS

1st Semester

Rajiv Gandhi University

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MATHEMATICS

MA [Economics]

First Semester

MAECO - 404

RAJIV GANDHI UNIVERSITY

Arunachal Pradesh, INDIA - 791 112

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About the University

Rajiv Gandhi University (formerly Arunachal University) is a premier institution for higher education in the state of Arunachal Pradesh and has completed twenty-five years of its existence. Late Smt. Indira Gandhi, the then Prime Minister of India, laid the foundation stone of the university on 4th February, 1984 at Rono Hills, where the present campus is located.

Ever since its inception, the university has been trying to achieve excellence and fulfill the objectives as envisaged in the University Act. The university received academic recognition under Section 2(f) from the University Grants Commission on 28th March, 1985 and started functioning from 1st April, 1985. It got financial recognition under section 12-B of the UGC on 25th March, 1994. Since then Rajiv Gandhi University, (then Arunachal University) has carved a niche for itself in the educational scenario of the country following its selection as a University with potential for excellence by a high-level expert committee of the University Grants Commission from among universities in India.

The University was converted into a Central University with effect from 9th April, 2007 as per notification of the Ministry of Human Resource Development, Government of India.

The University is located atop Rono Hills on a picturesque tableland of 302 acres overlooking the river Dikrong. It is 6.5 km from the National Highway 52-A and 25 km from Itanagar, the State capital. The campus is linked with the National Highway by the Dikrong bridge.

The teaching and research programmes of the University are designed with a view to play a positive role in the socio-economic and cultural development of the State. The University offers Undergraduate, Post-graduate, M.Phil and Ph.D. programmes. The Department of Education also offers the B.Ed. programme.

There are fifteen colleges affiliated to the University. The University has been extending educational facilities to students from the neighbouring states, particularly Assam. The strength of students in different departments of the University and in affiliated colleges has been steadily increasing.

The faculty members have been actively engaged in research activities with financial support from UGC and other funding agencies. Since inception, a number of proposals on research projects have been sanctioned by various funding agencies to the University. Various departments have organized numerous seminars, workshops and conferences. Many faculty members have participated in national and international conferences and seminars held within the country and abroad. Eminent scholars and distinguished personalities have visited the University and delivered lectures on various disciplines.

The academic year 2000-2001 was a year of consolidation for the University. The switch over from the annual to the semester system took off smoothly and the performance of the students registered a marked improvement. Various syllabi designed by Boards of Post-graduate Studies (BPGS) have been implemented. VSAT facility installed by the ERNET India, New Delhi under the UGC-Infonet program, provides Internet access.

In spite of infrastructural constraints, the University has been maintaining its academic excellence. The University has strictly adhered to the academic calendar, conducted the examinations and declared the results on time. The students from the University have found placements not only in State and Central Government Services, but also in various institutions, industries and organizations. Many students have emerged successful in the National Eligibility Test (NET).

Since inception, the University has made significant progress in teaching, research, innovations in curriculum development and developing infrastructure.

About IDE

The formal system of higher education in our country is facing the problems of access, limitation of seats, lack of facilities and infrastructure. Academicians from various disciplines opine that it is learning which is more important and not the channel of education. The education through distance mode is an alternative mode of imparting instruction to overcome the problems of access, infrastructure and socio-economic barriers. This will meet the demand for qualitative higher education of millions of people who cannot get admission in the regular system and wish to pursue their education. It also helps interested employed and unemployed men and women to continue with their higher education. Distance education is a distinct approach to impart education to learners who remained away in the space and/or time from the teachers and teaching institutions on account of economic, social and other considerations. Our main aim is to provide higher education opportunities to those who are unable to join regular academic and vocational education programmes in the affiliated colleges of the University and make higher education reach to the doorsteps in rural and geographically remote areas of Arunachal Pradesh in particular and North-eastern part of India in general. In 2008, the Centre for Distance Education has been renamed as "Institute of Distance Education (IDE)."

Continuing the endeavor to expand the learning opportunities for distant learners, IDE has introduced Post Graduate Courses in 5 subjects (Education, English, Hindi, History and Political Science) from the Academic Session 2013-14.

The Institute of Distance Education is housed in the Physical Sciences Faculty Building (first floor) next to the University Library. The University campus is 6 kms from NERIST point on National Highway 52A. The University buses ply to NERIST point regularly.

Outstanding Features of Institute of Distance Education:

(i) At Par with Regular Mode

Eligibility requirements, curricular content, mode of examination and the award of degrees are on par with the colleges affiliated to the Rajiv Gandhi University and the Department(s) of the University.

(ii) Self-Instructional Study Material (SISM)

The students are provided SISM prepared by the Institute and approved by Distance Education Council (DEC), New Delhi. This will be provided at the time of admission at the IDE or its Study Centres. SISM is provided only in English except Hindi subject.

(iii) Contact and Counselling Programme (CCP)

The course curriculum of every programme involves counselling in the form of personal contact programme of duration of approximately 7-15 days. The CCP shall not be compulsory for BA. However for professional courses and MA the attendance in CCP will be mandatory.

(iv) Field Training and Project

For professional course(s) there shall be provision of field training and project writing in the concerned subject.

(v) Medium of Instruction and Examination

The medium of instruction and examination will be English for all the subjects except for those subjects where the learners will need to write in the respective languages.

(vi) Subject/Counselling Coordinators

For developing study material, the IDE appoints subject coordinators from within and outside the University. In order to run the PCCP effectively Counselling Coordinators are engaged from the Departments of the University, The Counselling-Coordinators do necessary coordination for involving resource persons in contact and counselling programme and assignment evaluation. The learners can also contact them for clarifying their difficulties in then respective subjects.

SYLLABI-BOOK MAPPING TABLE

Mathematics

Syllabi	Mapping in Book
UNIT I: Co-Ordinate Geometry (Two Dimensional) and Algebra Equation of a Straight Line: Slope, Intercept—Derivation of a Straight Line Given (a) Intercept and Slope and (b) Intercepts—Angle between Two Lines, Conditions of Line for Being Parallel. Circle: Derivation of the Equation of a Circle given a Point and Radius, Derivation of the Equation of a Parabola, Definition of Hyperbola and Ellipse, Binomial Expansion for a Positive, Negative or Fractional Exponent, Exponential and Logarithmic Series.	**Unit 1:** Coordinate Geometry and Algebra **(Pages 3-57)**
UNIT II: Matrix Algebra Scalar and Vector, Length of a Vector, Addition, Subtraction and Scalar Products of Two Vectors, Angle between Two Vectors, Cauchy-Schwarz Inequality, Vector Space and Normed Space, Basis of Vector Space, The Standard Basis, Spanning of Vector Space: Linear Combination and Linear Independence. Types of Matrices: Null, Unit and Idempotent Matrices, matrix Operations, Determinants, Matrix Inversion and Solution of Simultaneous Equations, Cramer's Rule, Rank of a Matrix, Characteristics Roots and Vectors.	**Unit 2:** Matrix Algebra **(Pages 59-135)**
UNIT III: Differentiation Limit and Continuity of Functions, Basic Rules of Differentiation, Partial and Total Differentiation, Indeterminate Form, L' Hopital Rules, Maxima and Minima, Points of Inflexion, Constrained Maximization and Minimization, Lagrangean Multiplier, Applications to Elasticity of Demand and Supply, Equilibrium to Consumer and Firm.	**Unit 3:** Differentiation **(Pages 137-201)**
UNIT IV: Integration Integral as Anti-Derivative, Basic Rules of Integration, Indefinite and Definite Integral, Beta and Gamma Functions, Improper Integral of $\int_0^\infty e^{-x^2}dx$, Application to Derivation of Total Revenue and Total Cost from Marginal Revenue and Marginal Cost, Estimation of Consumer Surplus and Producer Surplus, First Order Differential Equation.	**Unit 4:** Integration **(Pages 203-264)**
UNIT V: Linear Programming Concept, Objectives and Uses of Linear Programming in Economics, Graphical Method, Slack and Surplus Variables, Feasible Region and Basic Solution, Problem of Degeneration, Simplex Method, Solution of Primal and Dual Models.	**Unit 5:** Linear Programming **(Pages 265-311)**

CONTENTS

INTRODUCTION

NOTES

Mathematics is the study of quantity, structure, space and change. The mathematician, Benjamin Peirce called mathematics 'the science that draws necessary conclusions'. Hence, Mathematics is the most important subject for achieving excellence in any field of Science and Engineering. Mathematical statistics is the application of mathematics to statistics, which was originally conceived as the science of the state—the collection and analysis of facts about a country: its economy, land, military, population, and so forth. Mathematical techniques which are used for this include mathematical analysis, linear algebra, stochastic analysis, differential equations, and measure-theoretic probability theory.

Statistics is considered a mathematical science pertaining to the collection, analysis, interpretation or explanation and presentation of data. Statistical analysis is very important for taking decisions and is widely used by academic institutions, natural and social sciences departments, governments and business organizations. The word statistics is derived from the Latin word status which means a political state or government. It was originally applied in connection with kings and monarchs collecting data on their citizenry which pertained to state wealth, collection of taxes, study of population, and so on.

The subject of statistics is primarily concerned with making decisions about various disciplines of market and employment, such as stock market trends, unemployment rates in various sectors of industries, demographic shifts, interest rates, and inflation rates over the years, and so on. Statistics is also considered a science that deals with numbers or figures describing the state of affairs of various situations with which we are generally and specifically concerned. To a layman, it often refers to a column of figures or perhaps tables, graphs and charts relating to areas, such as population, national income, expenditures, production, consumption, supply, demand, sales, imports, exports, births, deaths, accidents, and so on.

This book, *Mathematics and Statistics*, has been designed keeping in mind the self-instruction mode format and follows a SIM pattern, wherein each unit begins with an 'Introduction' to the topic followed by the 'Unit Objectives'. The content is then presented in a simple and easy-to-understand manner, and is interspersed with 'Check Your Progress' questions to test the reader's understanding of the topic. 'Key Terms' and 'Summary' are useful tools for effective recapitulation of the text. A list of 'Questions and Exercises' is also provided at the end of each unit for effective recapitulation.

UNIT I: Co-Ordinate Geometry (Two Dimensional) and Algebra

1.1 Objectives

- To introduce with equations of straight line and circle.
- To provide some basic understanding of binomial expansion.
- To discuss about the basic concepts of vector.
- To discuss the matrix algebra.

1.2 Introduction

The coordinate geometry, vector, matrix algebra etc have enormous uses in the study of economics. Accordingly, this chapter aims to discuss the basics of Coordinate Geometry, Algebra and Matrix so that reader can have a preliminary understanding of the various topics covered. The distance between two points, equations of a straight line in various forms are discussed along with suitable examples. The equation of circle in standard form, binomial expansion etc are also included in the chapter. The chapter further introduces the concept of vector. Further, discussion on definition, types and operations of matrix, use of crammer's rule and characteristic (eigen) roots and vector are incorporated in the chapter.

1.3 Distance between two points

Let two points A(x_1, y_1) and B(x_2, y_2) and joints them by line AB. Again, draw AD and BE from points A and B to x-axis. By drawing $\perp AC\ to\ BE,$ we get the right-angled triangle ACB where $\angle ACB = 90^O$

Now, AC = DE = OE-OD

$\therefore$ AC $= \mathrm{x}_2 - \mathrm{x}_1$

And BC =EB-EC

= Y_2-Y_1

Now, as per Pythagoras Theorem,

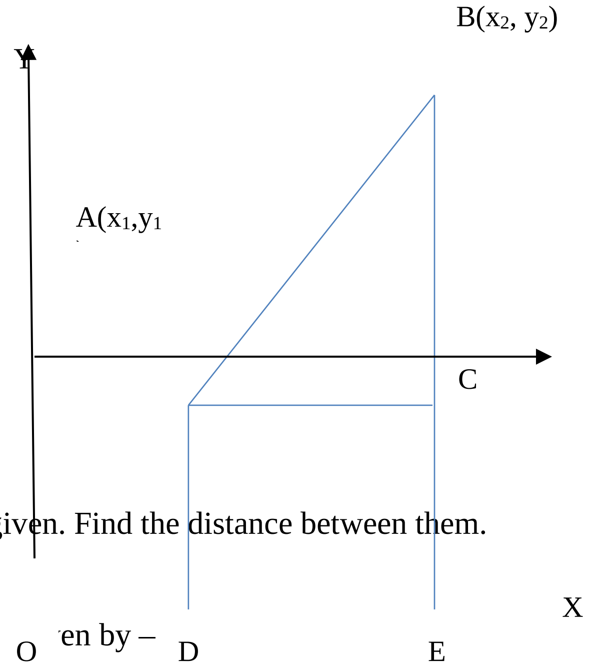
$AB^2 = AC^2 + BC^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\therefore AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore$ *Distance between points A and B is*

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Example: 1 Two points (6, - 8) and (2, -5) are given. Find the distance between them.

Solution

We know that the distance between two points is given by –

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(2-6)^2 + (-5-8)^2}$

$= \sqrt{16+9}$

$= \sqrt{25}$

$= 5$

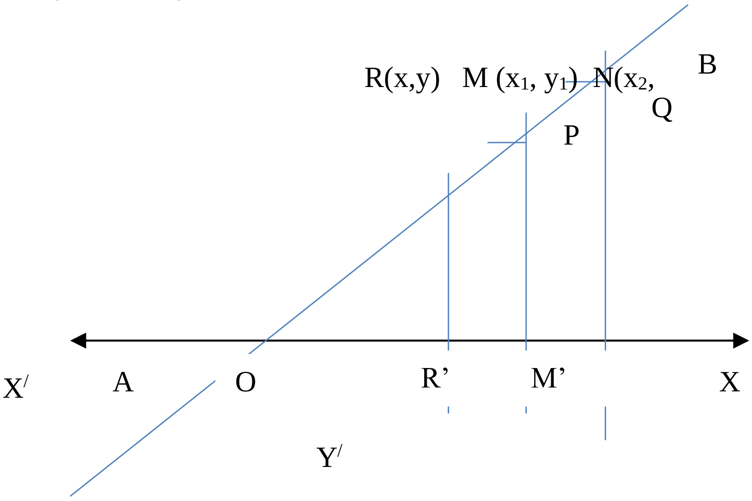
1.4 Equation of straight line

The simplest form of curve is straight line. Straight line is represented by a linear equation. Depending on the type of information available, the equation of straight line can be of different forms.

(a) Two points form

Let a straight line passes through M (x_1, y_1) and N (x_2, y_2) and R (x, y) be another point on it. Draw perpendiculars RR', MM' and NN' to x-axis. Again, draw perpendicular line RP to MM' and

MQ to NN'. Now we have to similar right angled triangles RPM and MQN



$\therefore \frac{MP}{RP} = \frac{NQ}{MQ}$

$Or \frac{M'M - M'P}{OM' - OR'} = \frac{NN' - N'Q}{ON' - OM'}$

$Or \frac{Y_1 - Y}{X_1 - X} = \frac{Y_2 - Y_1}{X_2 - X_1}$

$Or \frac{Y - Y_1}{X - X_1} = \frac{Y_2 - Y_1}{X_2 - X_1}$

$\therefore Y - Y_1 = \frac{Y_2 - Y_1}{X_2 - X_1}(X - X_1)$

is the equation of straight line

and the slope is given by $\left(\frac{Y_2 - Y_1}{X_2 - X_1}\right)$.

Example: 2 Given that (2, 2) and (4,8) are two points through which a straight line. Find the equation of the straight line and the slope.

$S\partial l^n$

The equation is

$y - 2 = \left(\frac{8-2}{4-2}\right)(x - 2)$

$\Rightarrow y - 2 = 3(x - 2)$

$\Rightarrow y - 2 = 3x - 6$

$\therefore y = 3x - 4$

Again the slope of the straight line is

$\frac{Y_2 - Y_1}{X_2 - X_1}$

$= \frac{8-2}{4-2}$

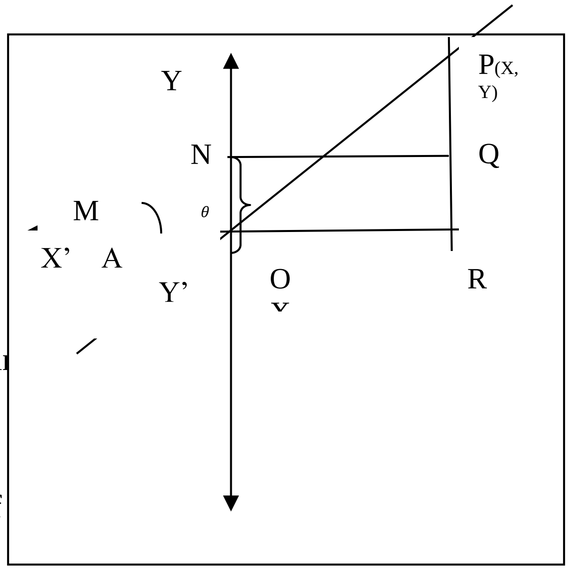
$= 3$

(b) Slope intercept form

Suppose intercept on Y-axis is c units and the angle of the straight line with X-axis is θ. Now draw perpendicular PR to X-axis and NQ to PR. Now we have PQN right angled triangle and

$\angle PNQ = \angle PMR = Q$

In right angled triangle PQN,



$tan\theta = \frac{PQ}{NQ}$

$= \frac{PR - QR}{NQ}$

$= \frac{y - c}{x}$ $As\ QR = NO = C$

$or\ xtan\theta = y - c$

$\therefore Y = xtan\theta + c$

Denoting numerical value of $tan\theta$ by m,

y = mx+c which is the equation of a straight line i

<u>Note:</u>

If c=0, y = mx, the line will pass through origin

If m= 0, y= c, the line will parallel to the x-axis.

Example 3: Write the equation of a straight line if

Solution:

The equation of the straight line is

y=mx + c

$or\ y = -\frac{5}{8}x - 5$

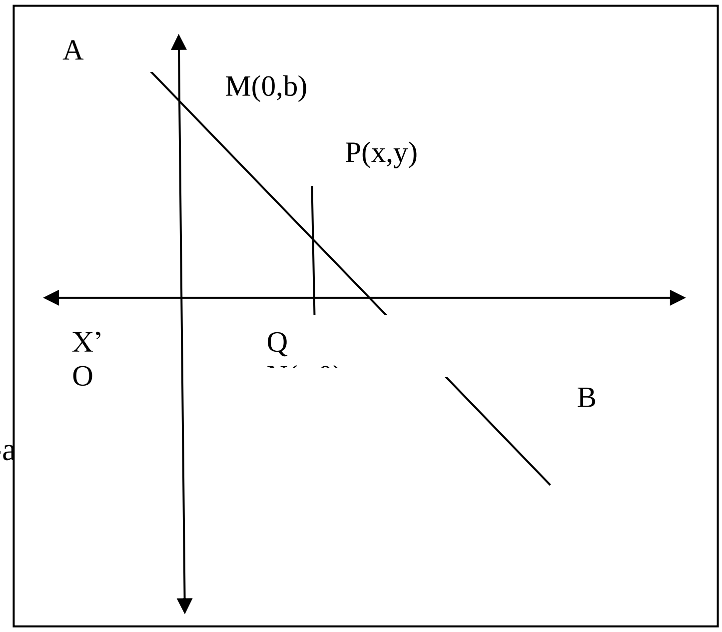
Or 8y = -5x-40

Or 5x + 8y + 40 = 0

(c) Intercepts form

Let a straight line AB cuts x – axis at N and y-axis at M such that ON= a and OM= b. Again, let P(x,y) be a point on AB and draw perpendicular PQ to x-axis.

Now we have similar triangle NOM and NQP



$\therefore \frac{PQ}{MO} = \frac{NQ}{ON}$

$or\ \frac{y}{b} = \frac{ON-OQ}{ON}$

$or\ \frac{y}{b} = \frac{a-x}{a}$

$or\ \frac{y}{b} = 1 - \frac{x}{a}$

$\therefore\ \frac{x}{a} + \frac{y}{b} = 1$

is the equation of a straight line in intercepts form.

Example : 4 : Find the intercepts on x-a

Solution:

2x-4y=3

$\Rightarrow {}^{2}/_{3}\, x - {}^{4}/_{3}\, y = 1$

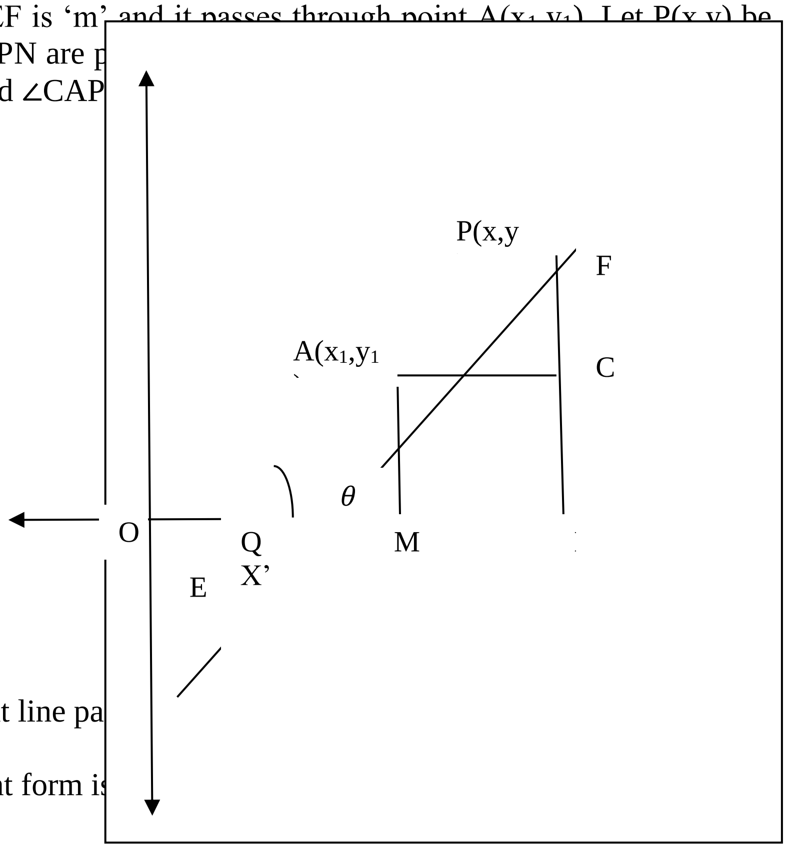
$\Rightarrow \frac{x}{{}^{3}/_{2}} - \frac{y}{{}^{3}/_{4}} = 1$

$\therefore Intercept\ on\ x-axis\ is\ {}^{3}/_{2}\ and\ on\ \ y-axis\ is - {}^{3}/_{4}$

(d) Slope point form

Suppose slope of the straight line EF is 'm' and it passes through point $A(x_1,y_1)$. Let P(x,y) be any point on the straight line EF. AM and PN are p

Now we have right angled triangle PCA and ∠CAP



$\therefore tan\theta = \frac{PC}{AC}$

$= \frac{PN-NC}{ON-OM}$

$= \frac{Y-Y_1}{X-X_1}$

Or $m = \frac{Y-Y_1}{X-X_1}$ as $tan\theta$= m

$\therefore y - y_1 = (x - x_1),$

is the equation of a straight line in slope point form.

Example: 5 Find the equation of the straight line pa

Solution:

The equation of a straight line in slope point form is

$y - y_1 = m(x - x_1)$

or y - 5 =6(x - 4)

or y - 5 = 6x - 24

or y =6x - 19

1.5 Angle between two straight lines.

Let two straight lines are given by y = m_1x + c_1 and y = m_2x + c_2. If the angle between the lines is 'θ',

$$tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Now, if $m_1 = m_2$ and $c_1 = c_2$ lines will coincide
if $m_1 = m_2$ but $c_1 \neq c_2$ lines are parallel and
If $m_1 \, m_2 = -1 \left(m_1 = \frac{-1}{m_2}\right)$, lines are perpendicular.

Example 6: Find the angle between lines x+2y=5 and 3x+y-11=0
Solution:

$x + 2\,y = 5$

$\Rightarrow 2y = -x + 5$

$\Rightarrow y = -{}^{1}/_{2} + {}^{5}/_{2}$

$\therefore \; m_1 = -{}^{1}/_{2}$

Again, $3x + y - 11 = 0$

$\Rightarrow y = -3x + 11$

$\therefore m_2 = -3$

$$\therefore tan\theta = \left|\frac{-3 - (-{}^{1}/_{2})}{1 + (-3)(-{}^{1}/_{2})}\right|$$

$$= \left|\frac{-3 + \frac{1}{2}}{1 + \frac{3}{2}}\right|$$

$$= \left|\frac{-{}^{5}/_{2}}{{}^{5}/_{2}}\right|$$

$= 1$

$\therefore \theta = 45^0$

1.6 The circle and its equation in standard form

A circle is the locus of a point that moves in a place such that the distance from the centre is always fixed. To find out the equation of a circle, let the centre C(h,k) and radius 'r'. Let draw a circle with centre C(h,k)and radius 'r'. Suppose P(x,y) be a point on the circle and draw PN perpendicular to x-axis and CM perpendicular to PN.

Now, CMP is a right angled triangle where,

CP= r

Again CM=QN=ON-OQ=X-h and

PM=PN-NM=Y-k

Now, by Pythagorean theorem,

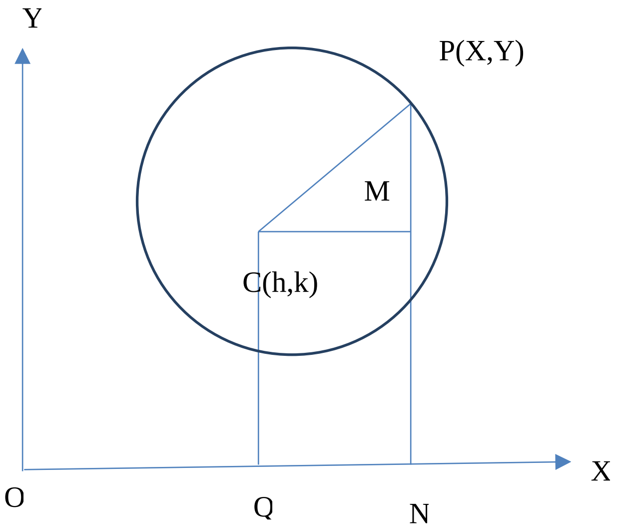
$(CP)^2 = (CM)^2 + (PM)^2$

$or\ \mathrm{r}^2 = (X - h)^2 + (Y - k)^2$

$or\ X^2 + h^2 - 2Xh + Y^2 + k^2 - 2Yk = \mathrm{r}^2$

$or\ X^2 + Y^2 + h^2 + k^2 - 2Xh - 2Yk = \mathrm{r}^2$

$is\ the\ required\ equation$



Example 7: Find the equation of the circle with centre (-1,2) and radius 3 unit.

Solution: The equation of the circle with centre (-1,2) and radius 3 units is

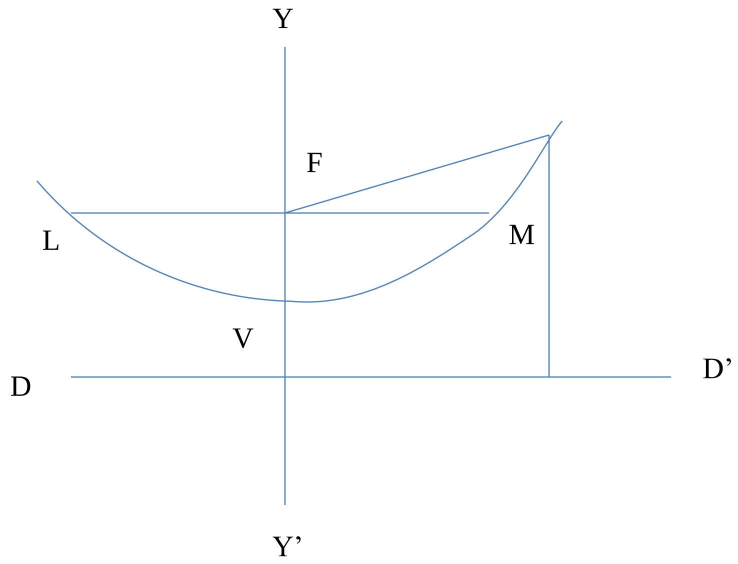
$$\{(X-(-1)^2\}+(Y-2)^2=3^2$$

$$\Rightarrow X^2+1+2X+Y^2+4-4Y=9$$

$$\Rightarrow X^2+Y^2+2X-4Y=9-5$$

$$\Rightarrow X^2+Y^2+2X-4Y=4$$

$$\Rightarrow X^2+Y^2+2X-4Y=2^2$$



1.7 Conic sections

A conic section is the locus of a point which moves in a plane in a way that the distance from a fixed point bears a constant ratio to the distance from a fixed line. The fixed point is focus, the fixed line is directrix and the constant ratio is eccentricity which is equal to 1. The straight line passes through the focus and perpendicular to the directrix is known as axis.

<u>**Parabola:**</u> a parabola is the locus of points in a plane that are equidistant from both directrix and focus.

Here, F – focus
DD’ – directrix
LM – latus rectum
N – Vertex
YY’ – axis of symmetry

The general form of the equation of a parabola is:

y = a(x-h)^2 + k or

x = a(y-k)^2 +h,

where (h,k) denotes the vertex.

The standard forms of the equation of parabola are:

y^2 = 4ax

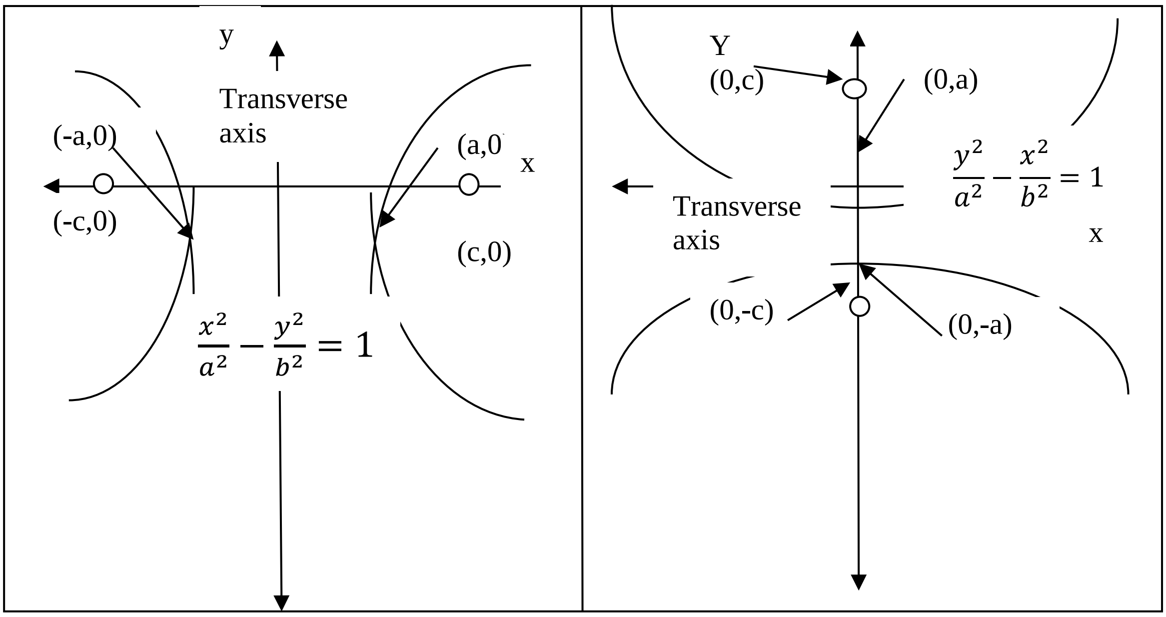
y^2 = -4ax

x^2 = 4ay

x^2 = -4ay

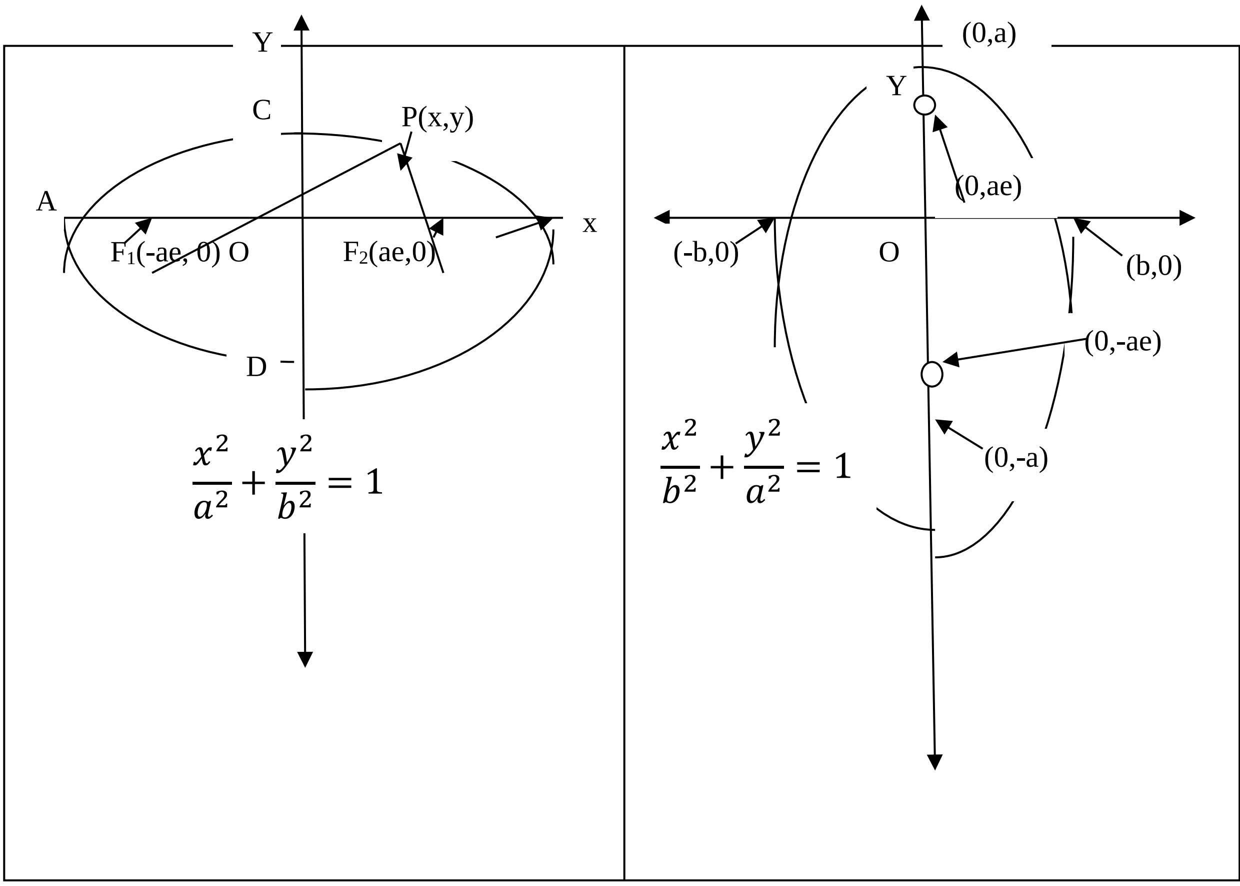
<u>Hyperbola</u>: It is the locus of a point that moves in a way that the distance from a fixed point called focus is always ‘e’ times its distance from directrix where e>1. In other words, a hyperbola is the set of all points in a plane whose distances form a fixed point in the plane, i.e, focus bears a constant ratio, greater than 1, to their distance from a fixed line in the plane,i.e, directrix. The standard form of the equation of hyperbola with centre at the origin is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ or } \frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$



Ellipse: An ellipse is the set of points in a plane the sum of whose distances from two fixed points (foci) is constant. Alternatively, it is the locus of a point which moves in such a way that the distance from the focus is always 'e' times of its distance from directrix where e<1. The equations of ellipse in standard form are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



n by

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$(x+a)^m = m_{c_0}x^m a^0 + m_{c_1}x^{m-1}a + m_{c_2}x^{m-2}a^2 + m_{c_3}x^{m-3}a^3 + \ldots + m_{c_m}x^0 a^m$

$= x^m + m_{c_1}x^{m-1}a + m_{c_2}x^{m-2}a^2 + m_{c_3}x^{m-3}a^3 + \ ... + a^m$……………………………..(3)

Multiplying (3) by $(x+a)$

$(x+a)(x+a)^m \ = (n+a)(x^m + m_{c_1}x^{m-1}a + m_{c_2}x^{m-2}a^2 + m_{c_3}x^{m-3}a^3 + \cdots + a^m)$

$or\ (x+a)^{m+1} = x^{m+1} + (m+1)_{c_1}x^m a + (m+1)_{c_2}x^{m-1}a^2 + \ \dots\dots\dots\dots\dots\dots.. + a^{m+1}$ as per equ…(3)

Assuming m+1 = n,

$(x+a)^n = x^n + n_{c_1}x^{n-1}a + n_{c_2}x^{m-2}a^2 + \ \dots\dots\dots\dots\dots + a^n$

Example 7: Expand $(2x+3)^5$

Solution:

Here, n= 5

∴ Number of terms in the expansion will be 6

$(2x+3)^5 = (2x)^5 + 5_{c_1}(2x)^4 3 + 5_{c_2}(2x)^3 3^2 + 5_{c_3}(2x)^2 3^3 + 5_{c_2}(2x)3^4 + 3^5$

$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$

<u>General Term:</u>

The general term for the expansion $(x+a)^n$ is given by

$T_{r+1} = n_{c_r}x^{n-r}a^r$ where T_{r+1} is the $(r+1)^{th}$ term.

Example 8:

Find the general term for the expansion $({}^{x}/_{2} - 2y)^6$

Solution

In the expansion $({}^{x}/_{2} - 2y)^6$,

We have $x = {}^{n}/_{2}, a = (-2y)\ and\ n = 6$

$\therefore T_{r+1} = \ 6_{c_r}({}^{x}/_{2})^{6-r}(-2y)^r$

$= (-1)^r 6_{c_r} x^{6-r}({}^{1}/_{2})^{6-r}(2y)^r$

$= (-1)^r 6_{c_r} x^{6-r} y^r \frac{2^r}{2^{6-r}}$

$= (-1)^r 6_{c_r} x^{6-r} y^r 2^{2r-6}$

<u>Middle term</u>

For the expansion $(x+a)^n$, we will have one middle term if ‘n’ is even and it will be $\left(\frac{n+2}{2}\right)^{th}$ term.

If ‘n’ is odd we will have two middle terms which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms.

Example 9: Find the middle term of the expansion $(3x-2)^{10}$

Solution:

In the expansion$(3x-2)^{10}$, n=10 is even

∴ There is one middle term and it is $\left(\frac{10+2}{2}\right)$ =6^{th} term

Now, the 6^{th} term is

$T_{5+1} = \ 10_{c_5}(3x)^5(-2)^5$ (Following the formula for general term)

$= (-1)^5 10_{c_5}(3x)^5\ 2^5$

$= -\ \frac{10!}{5!\,5!} 3^5 2^5 x^5$

$= -195055x^5$

<u>Exponential Series:</u>

The sum of the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is defined as e^x, i.e.,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

If x=1, e^1=e = 2.71828… which is an irrational number.

Now

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\therefore e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$or\ e = 1 + 1 + s_0$, where $S_0 = \frac{1}{2!} + \frac{1}{3!} + \cdots$

$\therefore e > 2$ as S_0 is positive…………………………………………………………………(1)

Again, we know that $n! > 2^{n-1}$ for all n>2

Therefore, if n=3, $\frac{1}{3!} < \frac{1}{2^2}$

n=4, $\frac{1}{4!} < \frac{1}{2^3}$

n=5, $\frac{1}{5!} < \frac{1}{2^4}$

Thus, $\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots < \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$

Or $\left(1 + \frac{1}{1!} + \frac{1}{2!}\right) + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots < \left(1 + \frac{1}{1!} + \frac{1}{2!}\right) + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$

Or e $< 1 + (1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots)$

Now, $a + r + r^2 + r^3 + r^4 + \cdots = \frac{a}{1-r}$ where ‘a’ is initial term and ‘r’ is common ratio.

Therefore, $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$ where 1 is initial term and $\frac{1}{2}$ is common ratio.

So, e $< 1 + 2$ ……………………………………………………………………………(2)

$\therefore from\ (1) and\ (2), 2 < e < 3$

Thus, ‘e’ lies between 2 and 3

<u>Logarithmic series:</u>

We know that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots, \qquad |x| < 1$$

$$\therefore \int_0^x \frac{1}{1+x} dx = x - {x^2}/{2} + {x^3}/{3} - {x^4}/{4} + \ldots\ldots\ldots \qquad |x| < 1$$

However,

$$\int_0^x \frac{1}{1+x} dx = \log(1+x)$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Similarly, $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

NIT II: Matrix Algebra

2.1 Vector

Vector vs Scalar:

Scalar is a quantity having only magnitude but no direction. For example, weight on the other hand, vector is a quantity having both magnitude and direction. For example, speed.

Definition of a vector:

Let P_1, P_1, …..P_n be any 'n' real numbers and P is ordered set of these real numbers such as P= (P_1, P_1, …..P_n), then P is a vector of order 'n'. Here, P_1, P_1, …..P_n are called components of the vector P.

When components are written in row, it is known as row vector. If components are written in column, vector is known as column vector.

Null vector: If all the components of a vector are zero, vector are zero, vector is known as null vector and denoted by 'O'. For example O=(0,0,0)

Unit vector: If the ith component of a vector is unity and other components are zero, the vector is known as unit vector. For a unit vector with i-th component as unity, the order of the vector is 'i'. Thus, E_1=[1,0,0], E_2= [0,1,0], E_3=[0,0,1,0].

Sum vector: If all the components of a vector are unity, the vector is called sum vector and denoted by 'I'. For example, I=[1,1,……1]

Equality of vectors:

Two vectors of same order are said to be equal if their corresponding elements are equal.

Let $P = (p_1, p_2, p_3) and\ Q = (q_1, q_2, q_3)$

Then P=Q if and only if $p_1 = q_1, p_2 = q_2\ and\ p_{3=}q_3$

Addition/subtraction of vectors:

Suppose $P = (p_1, p_2, \dots \dots . p_n)$

Q=$(q_1, q_2, \dots \dots . q_n)$

$R = (r_1, r_2, \dots \dots . r_n)$

Now for $R = P \pm Q,$ ith component is computed as $r_i = \ p_i \pm q_i$

Thus $R = P \pm Q$

$= (p_1 \pm q_1, p_2 \pm q_2, \dots\ p_n \pm q_n)$

If P,Q and S are vectors of same order, They will follows-

(i) P+Q=Q+P (Commutative law)

(ii) (P+Q)+S=P+(Q+S) (Associative law)

(iii) P+(-P)=O (Zero vector)

Multiplication of vector by Scalar:

If $P = (p_1, p_2, \dots \dots . p_n)$ is a vector of order 'n' and θ is a scalar,

$$Q = \theta P = (\theta p_1, \theta p_2 \dots \theta p_n$$

Properties

If P and S are two vectors of same order,

(i) $\theta(P + S) = \theta P + \theta S$

(ii) $\theta(\gamma p) = (\theta\gamma)p,$ where θ and γ are scalars.

Scalar product/dot product/Inner product:

Suppose $P = (p_1, p_2, \dots \dots . p_n)$

Q=$(q_1, q_2, \dots \dots . q_n)$

$\therefore P.Q = p_1q_1 + p_2q_2 + \cdots + p_nq_n$

$= \sum_{i=1}^{n} p_i q_i$

If dot product of two vectors is equal to zero, they are perpendicular to each other

Magnitude of a vector:

For a vector $P = (p_1, p_2, \dots \dots . p_n)$

magnitude/length is given by

$$|P| = \sqrt{{p_1}^2 + {p_2}^2 + \cdots + {p_n}^2}$$

<u>Linear combination of a set of vectors:</u>

A vector η is said to be a linear combination of the set $(\eta_1, \eta_2, \ldots \eta_r)$ if η can be expressed as $\eta = k_1\eta_1 + k_2\eta_2 + \cdots + k_r$, where $k_1, k_2, \ldots k_r$ are scalars not all zero.

<u>Linear dependence/Independence of vectors</u>

Vectors P_1, P_1, …..P_n are linearly independent iff

$$\sum_{j=1}^{n} \theta_j P_j = O, \;\; \theta_j = O, j = 1,2, \ldots n$$

If θ_j are not all zero but $\sum_{j=1}^{n} \theta_j P_j = O,$ vectors are linearly dependent.

Example 10: Examine whether vectors $a_1(3,2)$ and $a_2(1,4)$ are linearly dependent or independent.

<u>Solution:</u>

Let

$\theta_1(3,2) + \theta_2(1,4) = (0,0)$

Or $3\theta_1 + \theta_2 = 0 \; and \; 2\theta_1 + 4\theta_2 = 0$

$\Rightarrow \theta_2 = -3\theta_1$

Putting $\theta_2 = -3\theta_1 \; in \;\; 2\theta_1 + 4\theta_2 = 0,$

$\Rightarrow 2\theta_1 - 12\theta_1 = 0$

$\Rightarrow -10\theta_1 = 0$

$\therefore \theta_1 = 0$

$\therefore \theta_2 = 0$

$\therefore$ a_1 and a_2 are linearly independent.

Example 11

Show that $n_1 = (2, 3, -1) \; and \; n_2 = (4,6,-2)$ are linearly dependent.

Let $k_1n_1 + k_2n_2 = 0$

$\Rightarrow k_1(2,3,-1) + \; k_2(4, 6, -2) = 0$

$\Rightarrow (2k_1, 3k_1, -k,) + (4k_2, 6k_2, -k_2) = 0$

$\Rightarrow 2k_1 + 4k_{2,} 3k_1 + 6k_2, k_{1-}2k_2) = (0,0,0)$

$\therefore 2k_1 + 4k_2 = 0, \quad 3k_1 + 6k_2 = 0, \quad k_1 - 2k_2 = 0$

$\Rightarrow k_1 = -2k_2, \qquad \Rightarrow k_1 = -2k_2, \qquad \Rightarrow -k_1 = 2k_2$

$\Rightarrow k_1 = -2k_2$

Now,

$-2n_1 + n_2$

$= -2(2,3,-1) + (4, \; 6, -2)$

$= (-4, -3, \; 2) + (4, \; 6, -2)$

$= (0,0,0)$

$\therefore n_1 \; and \; n_2$ are linearly dependent.

1.10 Matrix

A matrix is a rectangular array of elements. A matrix of m x n order is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mxn} \qquad \text{Where } a_{ij}s \; are \; elements$$

$m \rightarrow$ no. of rows

$n \rightarrow$ no. of columns

<u>**Types of Matrices**</u>

Square matrix: If the number of rows(m) and number of columns(n) of a matrix is same i.e. m=n, the matrix is known as square matrix. For example,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 3 & 7 & 5 \end{bmatrix}_{3x3}$$

Null matrix: If all the elements of a matrix are zero, it is called null matrix and denoted by O.
For example,

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2x2}$$

Identity matrix: It is a square matrix which diagonal elements are equal to '1' and off-diagonal elements are zero. An identity matrix is also known as unit matrix. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3}$$

Diagonal matrix : A matrix which off-diagonal elements are zero but has scalar elements in the principal diagonal, not necessarily equal, is known as diagonal matrix. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3x3}$$

Equality of matrices:
Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if $a_{ij} = b_{ij}$ for all i and j and they are of same size. For example

$$If\ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\ and\ B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Then A= B, but

$$if\ C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\ and\ D = \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix},$$

C ≠ D

Arithmetic operations of matrices:

<u>Addition/Subtraction of matrices</u>

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ can be added provided they have same order. The sum A+B is obtained by adding the corresponding elements in A and B.

Let, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\ and\ B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$\therefore A + B = \begin{bmatrix} (1+3) & (2+4) \\ (3+5) & (4+6) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$$

Similarly,

$$A - B = \begin{bmatrix} (1-3) & (2-4) \\ (3-5) & (4-6) \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$$

If A,B and C are matrices of same size,

(i) A+B=B+A (Commutative)

(ii) A± (B±C) = (A±B)±C (Associative)

(iii)$(A \pm B)' = A' \pm B'$

<u>Matrix Multiplication</u>

The multiplication of two matrices A and B in the form AB requires that the number of columns of A equals the number of rows of B.

Example 13:

If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}_{2x2}\ and\ B = \begin{bmatrix} 5 & 7 & 9 \\ 6 & 8 & 0 \end{bmatrix}_{2x3}$

$$\therefore AB = \begin{bmatrix} (1 \times 5 + 3 \times 6) & (1 \times 7 + 3 \times 8) (1 \times 9 + 3 \times 0) \\ (2 \times 5 + 4 \times 6) & (2 \times 7 + 4 \times 8) (2 \times 9 + 4 \times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 31 & 9 \\ 34 & 46 & 18 \end{bmatrix}_{2x3}$$

If matrix A is of order (m x n) and matrix B is of order (n x m), then the order of AB is (m x m).
In general AB ≠ BA
Properties:
(i) $I_mA=AI_n=A$ where, I_m & I_n are identity matrices
(ii) (AB)C=A(BC)
(iii) C(A+B) = CA+CB
(iv) (A+B)C=AC+BC
(v) α(AB) =(αA) B = A(αB), α is a scalar

Scalar Multiplication

If a matrix is multiplied by a number, all the elements are multiplied by it. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\therefore \leftthreetimes\, A = \leftthreetimes \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \leftthreetimes\, a_{11} & \leftthreetimes\, a_{12} \\ \leftthreetimes\, a_{21} & \leftthreetimes\, a_{22} \end{bmatrix}$$

Transpose of a matrix

If A is a matrix of order (m x n), the transpose of A, i.e., $A^/$ is obtained by interchanging rows and columns and the order of $A^/$ is (n x m).

Example 14:

$$\text{If, } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2x3},$$

$$\therefore A' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3x2}$$

Note

(i) $(A^/)^/ = A$
(ii) $(A \pm B)^/ = A^/ \pm B^/$

$(A + B + C)^/ = A^/ + B^/ + C^/$
(iii) $(AB)^/ = B^/A^/$

$(ABC)^/ = C^/B^/A^/$

Determinant of a matrix

Determinant of a matrix is the scalar number associated with a square matrix.

$$If\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$= a_{11}a_{22} - a_{21}a_{12}$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 15:

$$If\ A = \begin{bmatrix} 4 & 7 \\ 1 & 3 \end{bmatrix}\ find\ |A|$$

Solution:
$|A| = (4 \times 3) - (7 \times 1) = 12 - 7 = 5$

Example 16: Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

$$= 1\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix}$$

$$= (3 \times 4 - 2 \times 3) - 2(2 \times 4 - 2 \times 3) + 3(2 \times 3 - 3 \times 3)$$

$$= 6 - 2(2) + 3(-3)$$

$$= 6 - 4 - 9$$

$$= 6 - 13$$

$$= -7$$

<u>Singular matrix and non-singular matrix:</u>

If determinant of a matrix is zero, it is known as singular matrix. Matrix A is said to be singular if |A|=0. For example,

$If\ A = \begin{vmatrix} 1 & 2 \\ 5 & 10 \end{vmatrix}$

$\therefore |A| = 10 - 10 = 0$

$\therefore A\ is\ a\ singular\ matrix$

If determinant of a matrix is non-zero, matrix is non-singular. Matrix B is said to be non-singular if |B|≠0

For example

$B = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\ \therefore |B| = 4 - 6 = -2 \neq 0$

$\therefore B\ is\ a\ non-singular\ matrix$

Note:

If A is a square matrix of order ‘n’, -

(i) If rank of A is ‘n’, A is non-singular and there is no linear dependence between rows or columns.
(ii) If rank of A is less than ‘n’, A is singular and there is linear dependence between at least two rows or column.

<u>Properties of determinant:</u>

(i) If all the elements of a row or column are zero, determinant of the matrix is zero.
(ii) $|A| = |A^{/}|$
(iii) If a new matrix B is obtained by interchanging any two rows or column of matrix A, the determinant values of new matrix will be same with the determinant value of original matrix but the sign will change, i.e., |B|= -|A|.
(iv) If two rows or columns of a matrix are multiple of one another, the determinant is equal to zero.
(v) If a scalar of times a column (or row) is added to another column (or row), the value of determinant of the matrix remains same.
(vi) If all the elements of a row or column of a determinant is multiplied by a scalar ⋋ , the value of the determinant is also multiplied by ⋋.
(vii) If A and B are two square matrix of same order, thus, |AB|=|A||B|.

Rank of a matrix

Rank of a matrix, i.e., ρ is the maximum number of linearly independent rows or columns in the matrix. For example

$$If\ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, |A| = 4 - 6 = -2 \neq 0$$

$$\therefore \rho(A) = 2$$

$$If\ B = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 0 & 6 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\therefore |B| = 2(0-12) - 5(6-6) + 4(6-0)$$

$$= -24 - 0 + 24$$

$$= 0$$

$\therefore \rho(B)$ is less than 3.

Let take the 2 x 2 sub-matrix of B. Suppose first sub-matrix is

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$$

Now $\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} = 0 - 15 = -15 \neq 0$

$\therefore\ \rho(B) = 2$

$$If\ C = \begin{bmatrix} 2 & 5 \\ 7 & 11 \\ 3 & 1 \end{bmatrix}$$

Now as maximum number of linearly independent rows (columns) must be equal to the maximum number of linearly independent rows (columns), $\rho(C)$ cann't be more than 2.

Let us take a (2x2) ordered sub-matrix

$$\begin{bmatrix} 2 & 5 \\ 7 & 11 \end{bmatrix}$$

Now

$$\begin{bmatrix} 2 & 5 \\ 7 & 11 \end{bmatrix} = 22 - 35 = -13 \neq 0$$

$\therefore \rho(C) = 2$

Minors and cofactors

Minors $(M_{ij})$of the element a_{ij} of matrix A is the determinant of the submatrix of A obtained by deleting i[th] row and j[th] column of original matrix A.

$$For\ matrix\ \ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minors are

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

A cofactor (C_{ij}) of the element (a_{ij}) of matrix A is the minor of the element with a prescribed sign such that

$$(C_{ij}) = (-1)^{i+j} M_{ij}$$

Example 17: Find minor and cofactor of the first row of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -3 & 7 \\ 1 & 2 & 4 \end{bmatrix}$$

<u>Solution:</u>

Minors of the first row are

$M_{11} = \begin{bmatrix} -3 & 7 \\ 2 & 4 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 6 & 7 \\ 1 & 4 \end{bmatrix}$, $M_{13} = \begin{bmatrix} 6 & -3 \\ 1 & 2 \end{bmatrix}$

= -12 - 14, = 24 - 7, = 12 + 3

= - 26, = 17, = 15

Cofactors of the first row are-

$C_{11} = (-1)^{1+1} \begin{bmatrix} -3 & 7 \\ 2 & 4 \end{bmatrix}$, $C_{12} = (-1)^{1+2} \begin{bmatrix} 6 & 7 \\ 1 & 4 \end{bmatrix}$, $C_{13} = (-1)^{1+3} \begin{bmatrix} 6 & -3 \\ 1 & 2 \end{bmatrix}$

=-26, =-17, = 15

<u>Adjoint of Matrix</u>

Adjoint of a matrix is the transpose of the cofactor matrix. For instance

If A $= \begin{bmatrix} 7 & 12 \\ 4 & 3 \end{bmatrix}$,

Cofactors are

$C_{11} = (-1)^{1+1}3$ $C_{12} = (-1)^{1+2}4$

$C_{21} = (-1)^{2+1}12$ $C_{22} = (-1)^{2+2}7$

$\therefore$ *cofactor matrix of A is*

$$\begin{bmatrix} 3 & -4 \\ -12 & 7 \end{bmatrix}$$

$\therefore$ *Adjoint of A* $= \begin{bmatrix} 3 & -4 \\ -4 & 7 \end{bmatrix}$

<u>Matrix Inversion</u>

Inverse of a matrix exist if the matrix is non-singular, i.e., the matrix do not have vanishing determinant. In other word, for matrix A, inverse will exist if |A|≠0. For matrix A, its inverse is given by

$$A^{-1} = \frac{1}{|A|}\, adj\ A$$

Some important properties of matrix inversion are –

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

(iii) $(A^{/})^{-1} = (A^{-1})^{/}$

Example 18

Find the inverse of matrix A if

$$A = \begin{bmatrix} 12 & -6 \\ -9 & 7 \end{bmatrix}$$

<u>Solution:</u>

Now $|A| = 12 \times 7 — (-6)(-9)$

$= 84 - 54$

$= 30 \neq 0$

As |A|≠0, matrix is non-singular and hence, inverse exist.

Now, cofactor matrix of A is

$$\begin{bmatrix} (-1)^{1+1}7 & (-1)^{1+2}(-9) \\ (-1)^{2+1}(-6) & (-1)^{2+2}(12) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 9 \\ 6 & 12 \end{bmatrix}$$

$$\therefore adj\ A = \begin{bmatrix} 7 & 6 \\ 9 & 12 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{30}\begin{bmatrix} 7 & 6 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} \frac{7}{30} & \frac{6}{30} \\ \frac{9}{30} & \frac{12}{30} \end{bmatrix}$$

Example 19

Solve the following system of equations by matrix inversion.

$4x_1+3x_2=28$

$2x_1+5x_2=42$

<u>Solution</u>

Let us express the given system of equation in matrix form as below-

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

$$or\ AX = B$$

$$where\ A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

$$or\ X = A^{-1}B$$

Now $|A|= 4x5 - 3x2 = 20 – 6 = 14\neq 0$

Cofactor matrix of A is

$$\begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}$$

$$\therefore adj\ A = \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & \frac{-3}{14} \\ \frac{-2}{14} & \frac{4}{14} \end{bmatrix}$$

$$\therefore\ X = A^{-1}B = \begin{bmatrix} \frac{5}{14} & \frac{-3}{14} \\ \frac{-2}{14} & \frac{4}{14} \end{bmatrix}\begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{14}(28) + \frac{-3}{14}(42) \\ \frac{-2}{14}(28) + \frac{4}{14}(42) \end{bmatrix}$$

$$= \begin{bmatrix} 10 - 9 \\ -4 + 12 \end{bmatrix}$$

Or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

$\therefore$ $x_1=1$ and $x_2=8$

<u>Cramer's Rule</u>

It is a method to solve a system of linear equations by using determinant. As per Cramer's rule

$$x_i = \frac{|A_i|}{|A|}$$

Where,

A is the coefficient matrix

A_i is the matrix obtained by replacing the column of coefficients of X_i with the column vector of constant.

Let a system of equations with 'n' equations and 'n' variables

$$a_{11}x_1 + a_{12}x_2 + - - - + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + - - - + a_{2n}x_n = b_2$$

$\vdots$
$a_{n1}x_1 + nx_2 + - - - + a_{nn}x_n \neq b_n$
As per Cramer's rule

$$x_i = \frac{|A_i|}{|A|}$$

$$where\ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{ and i=1. 2.,... n}$$

If $i = 1,\ x_1 = \frac{|A_1|}{|A|}\ and\ A_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Example 19: Solve the following using Cramer's rule-

$$6x_1 + 5x_2 = 49$$
$$3x_1 + 4x_2 = 32$$

<u>Solution:</u>
Let us express the given set of equations in matrix form as below-

$$\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 49 \\ 32 \end{bmatrix}$$

or $\text{AX} = \text{ B, Where A} = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} 49 \\ 32 \end{bmatrix}$

$$Now\ X_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 49 & 5 \\ 32 & 4 \end{vmatrix}}{\begin{vmatrix} 6 & 5 \\ 3 & 4 \end{vmatrix}} \frac{196 - 160}{24 - 15} = \frac{36}{9} = 4$$

$$X_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 6 & 49 \\ 3 & 32 \end{vmatrix}}{\begin{vmatrix} 6 & 5 \\ 3 & 4 \end{vmatrix}} \frac{192 - 147}{24 - 15} = \frac{45}{9} = 5$$

$\therefore x_1 = 4\ and\ x_2 = 5$

1.11 Characteristic (eigen) roots and vector

Let A is a square matrix and x is a nonzero vector. If there exists a scalar λ such that Ax=λx, x is said to be a characteristic vector of a square matrix A and λ is characteristic root of A.
Now Ax=λx

⇒Ax-λx=0
⇒(A-λ)x=0
⇒(A-λI)x=0, where I is an unit matrix

Here, λ is called characteristic roots or eigen value and x is called characteristic vector (eigen vector).

Let us construct a matrix [A-λI], where A =$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ which is a nonsingular square matrix, λ is a scalar and I is the identity matrix.

Now, A-λI =$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$- $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$=$\begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}$,

Here, A-λI is called characteristic matrix of A and $|A - \lambda I|$is characteristic determinant of A.

$|A - \lambda I|$= (2-λ)$(2 - \lambda) - 3.3$
= λ^2-4λ-5

$f(\lambda) = |A - \lambda I| = \lambda^2-4\lambda$-5 is called characteristic function of matrix A and $f(\lambda) = \lambda^2$-4λ-5 =0 is called characteristic equation of matrix A.

Here, $\lambda = -1$ and $\lambda = 5$ satisfies the characteristic equation which are called characteristic roots.
Example 20: Find characteristic vector and characteristic roots of the following matrix:

$$A= \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Solution

Let λ be the characteristic root.

Therefore, $A-\lambda I = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}$

Now, $|A-\lambda I| = (2-\lambda)(2-\lambda) - 3.3$

$= \lambda^2 - 4\lambda - 5$

Every characteristic root must satisfy

$f(\lambda) = |A-\lambda I| = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda^2 + \lambda - 5\lambda - 5 = 0$

$\Rightarrow \lambda(\lambda + 1) - 5(\lambda + 1) = 0$

$\Rightarrow (\lambda + 1)(\lambda - 5) = 0$

Therefore, $\lambda = -1$ and $\lambda = 5$ are the characteristic roots.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the characteristic vector

Now, $(A-\lambda I)x = 0$

$\Rightarrow \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

If $\lambda = -1$,

$\begin{bmatrix} 2-(1) & 3 \\ 3 & 2-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 3x_1 + 3x_2 \\ 3x_1 + 3x_2 \end{bmatrix} = 0$

$\Rightarrow 3x_1 + 3x_2 = 0$

$\Rightarrow 3x_1 = -3x_2$

$\Rightarrow x_1 = -x_2$

If $x_1=k$, $x_2=-k$

Therefore $x = \begin{bmatrix} -k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So, characteristic vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Again, If $\lambda = 5$,

$\begin{bmatrix} 2-(5) & 3 \\ 3 & 2-(5) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} -3x_1 + 3x_2 \\ 3x_1 - 3x_2 \end{bmatrix} = 0$

$\Rightarrow -3x_1 + 3x_2 = 0$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

If $x_1=k$, $x_2=k$

Therefore, $x = \begin{bmatrix} k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So, characteristic vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1.12 Summary

This chapter discusses the length of a straight line and equations of such line in four forms. The concepts circle, parabola, hyperbola and ellipse are discussed along with their equations in standard form. Binomial expansion is illustrated using with example. In the chapter, concept of vector, its types, scalar product, linear dependency/independency, etc are discussed. Definition of matrix, types, operations, determinants and its properties, matrix inversion, use of crammer's rule are some other topic covered in the chapter. Finally, characteristic roots and vector are explained.

1.13 Key terms

Vector: vector is a quantity having both magnitude and direction

Matrix: A matrix is a rectangular array of elements.

Determinant: Determinant of a matrix is the scalar number associated with a square matrix.

Rank of a matrix: Rank of a matrix is the maximum number of linearly independent rows or columns in the matrix.

Cramer's rule: It is a method to solve a system of linear equations by using determinant.

1.14 Questions

1. Derive the equation of a straight line in slope intercept form.
2. Derive the equation of a circle in standard form.
3. Expand $(2x-3)^5$ and find the middle term(s).
4. Compare vector and scalar.
5. What is a null vector?
6. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 8 & 9 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 2 & 1 \\ 5 & 5 & 2 \\ 2 & 3 & 1 \end{bmatrix}$, show that A+B= B+A.
7. Find transpose of $A = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 8 & 9 \\ 2 & 3 & 1 \end{bmatrix}$.
8. Discuss the properties of determinant.
9. Find the adjoint of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 4 \\ 1 & 3 & 5 \end{bmatrix}$.
10. Find the inverse of $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 3 & 2 & 0 \end{bmatrix}$.
11. Solve the following system of equations using crammer's rule:

    $6x_1+7x_2=56$

    $2x_1+3x_2=44$

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UNIT III: Diifferentiation

3.1 Introduction:

The study of effect of change(s) in certain economic variable(s) is a common economic phenomenon result of which the changes in the value of a related economic variable would make us know the director and magnitude of change in a particular economic variable. As for instance if the price of a product or price of related goods of the consumer changes then what will be the direction of magnitude of change in the demand for a commodity. Likewise due to change in the level of output, what would be total cost of production or what would be the change in the level of inputs employed. Therefore, it is very important to know the rate of change. The rate of change in the dependent variable with respect to the change in the independent variable, say, change in demand with respect to the change in price or change in the level of national income with respect to change in government expenditure or say parameter Play, marginal prosperity to consume. Therefore, the concept of differentiation or did I with you form the code of differential calculus that can be understood as a mathematical tool to find out the magnitude of direction of change in a particular variable due to change in the value of any other variable on a certain economic phenomenon.

3.2 Objectives:

The objectives of this module are to acquaint the learners about the concept of limit, rules of differentiation and their application in finding out rate of growth, marginal product, revenue, cost.

3.3 Limits:

The real number system would be incomplete without limits. They are essentially confined 2 days numbers death are calculated precisely in a finite number of steps, as for instance, play the case for integers and rational number.

Suppose, the equation $x^2-2=0$ has a song positive solution alpha=√2 & to provide arbitrarily accurate approximations to √2, We therefore, need to define √2 As a limit. So in in implicit may, we write √2~1.41421…This has an infinite sequence of play decimal expansion, i.e. 1,1.4, 1.41, 1.414, … which get closer to the limit √2. In which may, we can say √2 as a limit of a sequence of rational numbers, the same is true for irrational numbers, the same is true for irrational numbers aa well. Limits arise in the study of infinite series. The derivative of a function that measures rate of change of a variable with respect to the change of order related variable, can be defined using limits.

The derivative of a function in terms of limit is given by:

$$f'(a) = \lim_{h \to c} [f(a+h) - f(a)]/h$$

Thus, $\lim_{x \to a} f(x)$ can be read as when α approaches a constant quantity a from either side, there exists a definite finite number towards which f(x) tends. Such a numerical difference of f(x) as x tends to a.

Limits that do not exist:

Hypothetically, f is defined for all x close to a, but not necessarily at a. The function f(x) has the number A as its limits as x tends to a, provided that the number f(x) can be made as close to A as one nu …. For all x sufficiently close to (But not equal to) a.

$$\lim_{x \to c} f(x) = A \quad or \quad f(x) \longrightarrow A \quad as \quad x \longrightarrow a$$

Functions tending to infinity:

The behaviour of a function can be described as its argument tends to be Infinity large through marginal values.

Let, 'f' defined for arbitrarily large positive number x. So, f(x) has the limit A as X tends to infinity if f(x) can be made arbitrary close to A for all X sufficiently large.

$$\lim_{x \to \alpha} f(x) = A \quad or \quad f(x) \longrightarrow A \quad as \quad x \longrightarrow \alpha$$

Likewise,

$$\lim_{x \to \alpha} f(x) = B \; or \quad f(x) \longrightarrow B \quad as \quad x \longrightarrow \alpha$$

This indicates that f(x) can be made arbitrary close to B for all X which is sufficiently large and negative. Therefore, the horizontal asymptote for the graph of f as x tend to α is horizontal line y=A and y = B is a horizontal asymptote for the graph as x tend to α.

3.4 Continuity

A function is Continuous if small changes in the explanatory variable produce small changes in the function values. Play in geometric sense a function is continuous if its graph is connected i.e., it has no brakes. To make it simpler we can say that a function is continuous if its graphs drawn without lifting one's pencil of the paper. Otherwise, if the graph makes one or more jumps, we can therefore, say that f is discontinuous in form.

As for instance, f is defined on a domain that includes an open interval around a. Then f is continuous at x a provided that f(x) tends to f(a) in the limit as x tends to a:

f is continuous at x = a, if …

following are the continuous for f to be continuous at x=a

I) The function f must be defined/ expressed as…

II) The limit of f(x) as x-a must exist.

III) This limit must be equal to f(a).

Properties of continuous function

i. The sum or difference of two continuous function is a continuous function which is valid for any phoenix number of functions.
ii. The product of two continuous functions is a continuous function which is also valid for any finished number of functions.
iii. The quotientof two continuous function is a continuous function, provided the denominator is non zero anywhere for the range of values considered.
iv. If f(x) is continuous at x=a & f(a) is not equal to 0, then in the neighbourhood of x=a, f (x) has the same sign as that of f(a), i.e., we get a positive quantity & such that f (x) presence the same sign as that of f(a) for every value of f (x) in the interval ……….
v. If f(x) is continuous throughout the interval (a,b) & f(a) is not equal to f(b), here f (x) assumes every value between f(a) & f(b) at least once in the interval.
vi. A function is bounded which is continuous, throughout a closed interval.
vii. Continuous function attains its upper and lower bounds, at least once each in the interval.
viii. A function f(x) is considered to be continuous in a closed interval(a,b), therefore, attains every intermediate value at least once between its upper and lower bounds in the interval.

3.5 Basic Rules in differentiation

In any scientific discipline, the rate of change is very important to estimate the future of any particular variable (say demand for a commodity) or to predict the growth in population of a biologicalspecies in order to compute the future position of planet.

A derivative is defined as the central concept in mathematical analysis isthe rate of change of a function.

Most of the general rules were discovered independently of each other's by Isaac Newton(1642-1727) and Gottfried Leibriz (1646-1716).

Rate of change and derivative:

Suppose we have a general two variable function

Y=f(x)

Where, the value of y is associated with the value of x.

Therefore,the difference quotient of the above quotient function will be $\Delta y/\Delta x$, therefore, represents the rate of change in the value of y with respect to change in the value of x.It also represents the slope of the curve of the function y=f(x).

If we consider the function to be linear one, then

Y = f(x) = a+bx (a>0;b>0)

Where, b is the slope of the curve, we can then map y corresponding to the value of x in a diagram.

When $x = x_1$, then the corresponding value of y is
$Y_1 = a+bx_1$
Which represents the position A of a curve
Similarly, when $x = x_2$, then the corresponding value of y is $y = a+bx_2$ represents the position of a curve.
Therefore, when we move from A to Bon curve, the y value changes by Δy corresponding to the change in the value of x by Δx.
Therefore,
$\Delta y = y_1-y_2$
$\Delta y = bx_1-bx_2$
$\Delta y = b(x_1-x_2)$
$\Delta y = b\Delta x$
$\Delta y/\Delta x = b$, where b implies the slope of the curve.

BASIC RULES OF DIFFERENTIATION WITH SINGLE EXPLANATORY VARIABLE:
The rules of differentiation are derived for various forms offunctions using the definition of derivative.
1. Power function rule:
$y = f(x) = x^n$
The derivative of the above function is

$$\frac{dy}{dx} = f'x = nx^{n-1}$$

The power function rule of differentiationcan be generalised for
$\mathbf{y = f(x) = cx^n}$
Where c is a constant and n is a rational number.
Therefore, the differentiation of the above generalised power function

$$\frac{dy}{dx} = f'x = c\,nx^{n-1}$$

2. Constant Function Rule:
The function of Y such that
$Y = f(x) = c$
Where c is constant
The differentiation of the above function is always zero.

$$\frac{dy}{dx} = f'x = 0$$

This indicates that whenever the x value changes, the y value remains the same.An example of such situation is the constant price under perfect competitive market.
3. LOGARITHMIC FUNCTION RULE:
The function under logarithmic form is:
$y = f(x) = \log(x)$
Thus,

$$\frac{dy}{dx} = f'x = \frac{1}{x}$$

RULES OF DIFFERENTIATION FOR TWO OR MORE FUNCTIONS OF THE SAME EXPLANATORY FUNCTIONS:
In economic studies, we often experience certain situation, where a particular economic variable is observed as either a sum of two functions or difference of two functions or product of two functions or division of two functions of the same explanatory variable.As for example, profit can be expressed as the difference between total revenue andtotal cost function. Again, total revenue can be expressed as

the product of two functions,price and quantity i.e.TR=P*Q.Similarly, the Average Revenue can be expressed as quotient of two functions i.e., total revenue function divided by quantity sold{AR=TR/Q}.So here the rules of differentiation either for sum or difference of two functions,or for product of two functions or the quotient of two functions are not same.

1. Sum –Difference Rule:

A function which is the sum of two functions of the same independent variable, then the derivative of the sum of the functions is equal to the sum of the derivatives of the functions

$$y = f(x) = g(x) + h(x)$$

$$\frac{dy}{dx} = f'x = g'(x) + h'(x)$$

2. Quotient Rule of Differentiation:

Suppose the function is:

$$y = \frac{f(x)}{g(x)}$$

Therefore,

$$\frac{dy}{dx} = f'x = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

RULES OF DIFFERENTIATION FOR FUNCTIONS OF DIFFERENT VARIABLES:

1. Chain Rule of Differentiation:

Under chain rule, if we have a function y = f(u) where u is, in turn, a function of another variable x such that u = g(x), then the derivative of y with respect to x is the product of the derivative of y with respect to u and derivative of u with respect to x. Symbolically,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = f'(u) \times g'(x)$$

The above rule indicates that the effect of change in the value of x on the value of y is measured via the u variable. A given Δx will result in a corresponding Δu via u = g(x). In turn, Δu will result in corresponding Δy via y = f(x).

Example:

If $y=5u^2$ and $u=2+3x-x^2$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du}(5u^2) \times \frac{d}{dx}(2 + 3x - x^2)$$

$$\frac{dy}{dx} = 10u(3 - 2x)$$

$$\frac{dy}{dx} = 10(2 + 3x - x^2)(3 - 2x)$$

$$\frac{dy}{dx} = 10(6 - 4x + 9x - 6x^2 - 3x^2 + 2x^3)$$

$$\frac{dy}{dx} = 10(6 + 5x - 9x^2 + 2x^3)$$

2. Inverse Function Rule:

If a function y = f(x) represents one to one mapping such that not only a given value of x will yield a unique corresponding value of y, but also a given value of y will yield a unique value of x, then the function y = f(x) will have an inverse function

$$x = f^{-1}(y) = g(y) \quad \text{(say)}$$

For example, we can consider a demand function Q=f(P), where quantity demanded (Q) due to change in price (P) is inversely related implying that an increase in price will decrease demand. The relationship between Q and P is equally true in the form of price function P = g(Q) implying that an increase in quantity demanded will raise the price. Therefore, the function Q=f(P) has an inverse function.

$$P = f^{-1}(Q) = g(a)$$
$$x = f^{-1}(y) = g(y) \ \text{(say)}$$

In such situation:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Example: If y = 4x+50, then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{4}$$

This can be found out by making x as a function of y. so, the above function can be rewritten as:

$$x = \frac{y}{4} - 12.5$$
$$\text{So,} \frac{dx}{dy} = \frac{1}{4}$$

2.6 Partial and Total Differentiation:

In most of economic problems, what we observe is that a particular economic variable tends to depend on a number of other independent variables. Similarly, some of the independent variable, may also have a positive effect on the dependent variable while others have a negative effect on the dependent variable. As for instance, let us take the demand function where the Q_d is the quantity demanded and Q_d depends on the price of the product (P), Price of the related goods (P_R), income of the consumer (I) and size of the family (S), thus, Q_d= f(P, P_R, I, S).

Again, the demand for product tends tobe negative when there is a rise in the price of the product and the price of the complementary goods. But the on the other hand the condition such as increase in income of the consumer, increase in the size of the family of consumer will also raise the demand of the product. So, it is quite impossible to quantify the effect of an individual effect of the independent variable on the dependent variable. Therefore, in order to solve the such problems, we need to assume a particular independent variable changes but the other independent variables do not change at the same time. In order to quantify the effect of change in income on demand, price of the product and prices of the related goods and size of family of the consumer are assumed to be same. So, when we try to trace cut such effects by applying the technique of differentiation, then it is called as **partial derivatives**. But, on the other hand, when all the independent variables, which is termed as total differentiation.

1. Partial Differentiation:

Let us consider a function of 'n' independent variables–

$$Y = f(X_1, X_2, X_3, \dots, X_n)$$
$$Y + \Delta Y = f(X_1 + \Delta X_1, X_2, X_3, \dots, X_n)$$
$$\Delta Y = f(X_1 + \Delta X_1, X_2, X_3, \dots, X_n) - Y$$
$$\frac{\Delta Y}{\Delta X_1} = \frac{f(X_1 + \Delta X_1, X_2, X_3, \dots, X_n) - Y}{\Delta X_1}$$
$$\frac{\Delta Y}{\Delta X_1} = \frac{f(X_1 + \Delta X_1, X_2, X_3, \dots, X_n) - f(X_1 + \Delta X_1, X_2, X_3, \dots, X_n)}{\Delta X_1}$$

$\Delta Y/\Delta X_1$ represents the rate of change in Y with respect to a change in the value of X_1, taking other independent variables as constant. So, by taking the limiting value of X_1 equal to zero, the limit of the quotient $\Delta Y/\Delta X_1$is called the partial derivative of Y with respect to X_1.

The rules of partial differentiation are almost same as the rules for ordinary differentiation.

2. Total Differentials:

Considering a function Y=f(X) where due to an arbitrary change in the value of X, there is a corresponding change in the value of Y. So, as the difference quotient (ΔY/ΔX), the rate of change in Y is given as-

$$\Delta Y = \frac{\Delta Y}{\Delta X} \times \Delta X \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

The equation 1 here can be rewritten as-

$$dY = \frac{dY}{dX} \times dX$$

$$\text{or } \Delta Y = f'X \times dX \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Here, due to the infinitesimal change in the value of X, there has been an infinitesimal change in the value of Y, so the difference quotient tends to be $\frac{dY}{dX}$ and the infinitesimal change in X and Y are implicated as dX and dY respectively.

When there are functions of more than one explanatory variables, total differentials are applied, such that

$$dY = \left(\frac{\delta Y}{\delta X_2} \times dX_1\right) + \left(\frac{\delta Y}{\delta X_2} \times dX_2\right)$$

Given a function Y= f (X_1, X_2), when due to the infinitesimal change in both X and Y, there has been a total change in Y. This is called total differential of the Y function.Therefore, total differentiation can be found out via the process of function total differential.

APPLICATIONS ELASTICITY OF DEMAND AND SUPPLY:

The price elasticity is a measure of the responsiveness of demand to changes in the commodity’s own price. If the changes in price are very small, we use the measure of point elasticity of demand. If the change in the price is not so small then we use the elasticity of demand as a relative measure.

The point elasticity of demand is defined as the proportionate change in the quantity demanded resulting from a very small proportionate change in price.

$$E_p = \frac{\frac{dQ}{Q}}{\frac{dP}{P}}$$

$$Or, \quad E_p = \left(\frac{dQ}{dP}\right) \times \left(\frac{P}{Q}\right)$$

THE RELATIONSHIP BETWEEN MARGINAL REVENUE AND PRICE ELASTICITY:

The marginal revenue (MR) is related to the price elasticity of demand with the formula

$$MR = P\left(1 - \frac{1}{e}\right)$$

Such relationship is crucial for price theory.

Proof, the demand function is given as –

P= f(Q)

The Total Revenue (TR) is

TR=P X Q = [f(Q)] X Q

Thus, the MR is –

$$MR = \frac{d(PQ)}{dQ} = P \times \frac{d(Q)}{dQ} + Q \times \frac{d(P)}{dQ} = P + Q.\frac{dP}{dQ}$$

The price elasticity of demand is defined as –

$$E_P = -\frac{dQ}{dP} \times \frac{P}{Q}$$

$$E_P \times \frac{Q}{P} = -\frac{dQ}{dP}$$

$$E_P = -\frac{dQ}{dP} \times \frac{P}{Q}$$
$$E_P = -\frac{dQ}{dP} \times \frac{P}{Q}$$
$$-\frac{P}{QE_P} = -\frac{dP}{dQ}$$
Substituting dP/dQ in the expression of the MR,
$$MR = P + Q.\frac{dP}{dQ}$$
$$MR = P - Q.\frac{P}{E_P Q}$$
$$MR = P - \frac{P}{E_P}$$

RELATIONSHIP BETWEEN TOTAL REVENUE, MARGINAL REVENUE AND PRICE ELASTICITY:

If the demand curve is falling, the TR at first increases and reaches a maximum and then starts falling. The TR curve reaches its maximum level at the point where E_P=1, because at this point its slope, the marginal revenue is equal to zero, as
$$MR = P - \frac{P}{E_P}$$
$$MR = P - \frac{P}{1} = 0$$
If E_P> 1, the TR curve has a positive slope, i.e., it is still increasing and hence has not reached its maximum point as –
$P > 0$ & $\left(1 - \frac{1}{E_P}\right) > 0$, therefore, MR>0. If E_P<1 the TR curve has a negative slope, i.e., it is falling as $P > 0$ & $\left(1 - \frac{1}{E_P}\right) < 0$, therefore, MR< 0.

So, if the demand is inelastic (E_P< 1), a rise in price leads to a rise in total revenue and a fall in price leads to a fall in total revenue. If the demand is elastic (E_P>1), a rise in the price result in a fall of the total revenue, while a fall in price will result in a rise in the total revenue. On the other hand, if the demand has unitarily elastic, total revenue is not affected by changes in price.

EQUILIBIRUM OF THE CONSUMER:

In a sample model consisting of a single commodity say X and the consumer can either buy X or retain his money income Y. The consumer is in equilibrium when the marginal utility of X is equated to its market price (P_X).
$$MU_X=P_X$$
If the marginal utility of X is greater than its price, the consumer can increase his welfare by purchasing more units of X. If the marginal utility of X is less than its price, the consumer can cut down the quantity of X and keeping more of his income unspent. Therefore, he attains the maximization of his utility when $MU_X=P_X$.

In case, there are more commodities, the condition for the equilibrium f the consumer is the equality of the ratio of the marginal utilities of the individual commodities to their prices, as,
$$\frac{MU_X}{P_X} = \frac{MU_Y}{P_Y} = \cdots = \frac{MU_n}{P_n}$$

Mathematical Derivation:

The utility function–

$U = f(Q_X)$

Where, utility is measured in monetary units. If the consumer buys Q_X his expenditure is $Q_X P_X$ the consumer seeks to maximise the difference between his utility and his expenditure–

$U - P_X Q_X$.

The necessary condition for a maximum is that the partial derivative of the function with respect to Q_X be equal to zero.

$$\frac{\delta U}{\delta Q_X} - \frac{\delta(P_X Q_X)}{\delta Q_X} = 0$$

We can obtain,

$$\frac{\delta U}{\delta Q_X} = P_X \quad or \quad MU_X = P_X$$

The utility derived from spending an additional unit of money must be same for all commodities. The consumer derives greater utility from any other commodity, he can raise his welfare by spending more that particular commodity and less on other, until the above condition for equilibrium is satisfied.

2.7 Let us sum up

2.8 Keywords

2.9 Short questions

2.10 Long questions

2.11 Further/Suggested Readings

1. Allen, R.G.D., *Mathematical Analysis for Economists*, Macmillan, 1976.

2. Chiang, A.C., *Fundamental Methods of Mathematical Economics*, McGraw Hill, Kogakusha, New Delhi, 1974.

3. Yamane, T., *Mathematics for Economists*, Prentice Halls, New Delhi, Indian Reprint, 1986.
4. Mehta and Madnani, *Mathematics for Economists*, Sultan Chand & Sons, 2013.

UNIT 4 INTEGRATION

NOTES

Structure

4.0 INTRODUCTION

In this unit, you will learn about the basic rules of integral calculus. Integration is the reverse process of differentiation. When we do not give a definite value to the integral, then it referred to as indefinite integral, while when we give the lower and upper limits, it is referred to as definite integral. A definite integral is a function only of its limits and not of the variable which may be changed. If the limits of a definite integral are changed then the sign of the integral also changes. You will learn elementary methods and properties of integration, definite integral and its properties, and indefinite integral. You will also learn a few methods by which integration of rational and irrational functions can be performed.

This unit will also discuss about the applications of integral calculus in economics multiple integrals, and Fourier series.

NOTES

4.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Understand elementary methods and properties of integration
- Define the term definite integrals
- Explain properties of definite integrals
- Discuss the concept of indefinite integrals
- Apply integral calculus to find the length, area and volume
- Describe the significance of multiple integrals
- Find the Fourier series expansion
- Explain the applications of integration in economics

4.2 ELEMENTARY METHODS AND PROPERTIES OF INTEGRATION

After learning differentiation, we now come to the 'reverse' process of it, namely integration. To give a precise shape to the definition of integration, we observe: If $g(x)$ is a function of x such that,

$$\frac{d}{dx} g(x) = f(x)$$

then we define integral of $f(x)$ with respect to x, to be the function $g(x)$. This is put in the notational form as,

$$\int f(x)\, dx = g(x)$$

The function $f(x)$ is called the Integrand. Presence of dx is there just to remind us that integration is being done with respect to x.

For example, since $\frac{d}{dx} \sin x = \cos x$

$$\int \cos x\, dx = \sin x$$

We get many such results as a direct consequence of the definition of integration, and can treat them as 'formulas'. A list of such standard results are given:

(1) $\int 1\, dx = x$ because $\frac{d}{dx}(x) = 1$

(2) $\int x^n\, dx = \frac{x^{n+1}}{n+1}\ (n \neq -1)$ because $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1$

(3) $\int \frac{1}{x}\, dx = \log x$ because $\frac{d}{dx}(\log x) = \frac{1}{x}$

(4) $\int e^x dx = e^x$ because $\frac{d}{dx}(e^x) = e^x$

(4) $\int \sin x\, dx = -\cos x$ because $\frac{d}{dx}(-\cos x) = \sin x$

NOTES

(5) $\int \cos x\, dx = \sin x$ because $\frac{d}{dx}(\sin x) = \cos x$

(6) $\int \sec^2 x\, dx = \tan x$ because $\frac{d}{dx}(\tan x) = \sec^2 x$

(7) ($\int \text{cosec}^2 x\, dx = -\cot x$ because $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$

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)

(8) $\int \sec x \tan x\, dx = \sec x$ because $\frac{d}{dx}(\sec x) = \sec x \tan x$

(9) ($\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x$ because $\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x$

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0
)

(10) ($\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x$ because $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

1
1
) $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x$ because $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(11) ($\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x$ because $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

1
2
)

(12) (
1
3
)

(13) $\int \frac{1}{ax+b}\, dx = \frac{\log(ax+b)}{a}$ because $\frac{d}{dx}\left[\frac{\log(ax+b)}{a}\right] = \frac{1}{ax+b}$

(14) ($\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{n+1} \cdot \frac{1}{a} \quad (n \neq -1)$

1
5
)

because $\frac{d}{dx}\frac{(ax+b)^{n+1}}{a(n+1)} = (ax+b)^n,\ n \neq -1$

(15) $\int a^x\, dx = \frac{a^x}{\log a}$ because $\frac{d}{dx} a^x = a^x \log a$

One might wonder at this stage that since

$$\frac{d}{dx}(\sin x + 4) = \cos x$$

Then, by definition, why $\int \cos x\, dx$ is not $(\sin x + 4)$? In fact, it could very well have been any constant. This suggests perhaps a small alteration in the definition.

We now define integration as:

If $\dfrac{d}{dx}g(x) = f(x)$

Then, $\int f(x)\, dx = g(x) + c$

Where c is **some** constant, called the *constant of integration.* Obviously, c could have any value and thus, integral of a function is not unique! But, we could say one thing here, that *any two integrals of the same function differ by a constant.* Since c could also have the value zero, $g(x)$ is one of the values of $\int f(x)dx$. By convention, we will not write the constant of integration (although it is there), and thus, $\int f(x)dx = g(x)$, and our definition stands.

The above is also referred to as **Indefinite Integral** (indefinite, because we are not really giving a definite value to the integral by not writing the constant of integration). We will give the definition of a definite integral later.

NOTES

4.2.1 Some Properties of Integration

The following are the some properties of integration:

(i) *Differentiation and integration cancel each other.*

The result is clear by the definition of integration.

Let $\frac{d}{dx}\ g(x) = f(x)$

Then, $\int f(x)\ dx = g(x)$ [By definition]

$\Rightarrow\quad \frac{d}{dx}\left[\int f(x)\, dx\right] = \frac{d}{dx}\,[g(x)] = f(x)$

Which proves the result.

(ii) *For any constant a,* $\int a\ f(x)\, dx\ =\ a\int f(x)\, dx$

Since $\frac{d}{dx}\left(a\int f(x)dx\right) = a\frac{d}{dx}\int f(x)\ dx$

$= a\,f(x)$

By definition, $\int a\ f(x)\ dx\ = a\int f(x)\ dx$

(iii) *For any two functions* $f(x)$ and $g(x)$,

$$\int [\,f(x)\ \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

As $\frac{d}{dx}\left[\int f(x)\, dx \pm \int g(x)dx\right] = \frac{d}{dx}\int f(x)\ dx \pm \frac{d}{dx}\int g(x)\ dx$

$= f(x) \pm g(x)$

It follows by definition that,

$$\int [\,f(x) \pm g(x)]dx = \int f(x)\, dx \pm \int g(x)dx$$

This result could be extended to a finite number of functions, i.e., in general,

$$\int [\,f_1(x) \pm f_2(x)\ \pm \ldots \pm f_n(x)]dx = \int f_1(x)\ dx \pm \int f_2(x)dx\ \pm \ldots\ \pm \int f_n(x)\ dx$$

Example 4.1: Find $\int (2x-3)^2\ dx.$

Solution: We have,

$$\int (2x-3)^2\ dx = \int (4x^2\ + 9 - 12x)\ dx$$

$$= \int 4x^2\ dx + \int 9\ dx - \int 12x\ dx$$

$$= 4\int x^2 dx\ +\ 9\int dx\ -\ 12\int x dx$$

$$= 4\frac{x^3}{3}\ + 9x - \frac{12x^2}{2}$$

$$= \frac{4}{3}\ x^3 - 6x^2 + 9x$$

Example 4.2: Find $\int (2x+1)^{1/3}\, dx$.

Solution: We have,

$$\int (2x+1)^{1/3}\, dx = \frac{(2x+1)^{1/3+1}}{\frac{1}{3}+1} \times \frac{1}{2}$$

$$= \frac{3}{8}(2x+1)^{4/3}$$

Example 4.3: Solve $\int \frac{x^3}{x+1}\, dx$.

Solution: By division, we note

$$\frac{x^3}{x+1} = (x^2 - x + 1) - \frac{1}{x+1}$$

Thus, $$\int \frac{x^3}{x+1}\, dx = \int (x^2 - x + 1)\, dx - \int \frac{1}{x+1}\, dx$$

$$= \int x^2 dx - \int x dx + \int dx - \int \frac{1}{x+1}\, dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log(x+1)$$

Example 4.4: Find $\int \sqrt{1+\cos 2x}\, dx$

Solution: We observe,

$$\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{2\cos^2 x}\, dx = \sqrt{0} \int \cos x\, dx$$

$$= \sqrt{2}\sin x$$

NOTES

4.2.2 Methods of Integration

The following are the methods of integration:

To evaluate $\int \frac{f'(x)}{f(x)}\, dx$ where $f'(x)$ is the derivative of $f(x)$

Put $f(x) = t$, then $f'(x)dx = dt$

Thus, $\int \frac{f'(x)}{f(x)}\, dx = \int \frac{dt}{t} = \log t = \log f(x)$

To evaluate $\int [f(x)]^n\, f'(x)dx,\ n \neq -1$

Put $f(x) = t$, then $f'(x)dx = dt$

Thus, $\int [f(x)]^n\, f'(x)dx \quad \int t^n\, dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1}$

To evaluate $\int f'(ax+b)dx$

Put $ax + b = t$, then, $adx = dt$

$$\int f'(ax+b)dx = \int f'(t)\frac{dt}{a} = \frac{1}{a}\int f'(t)\, dt = \frac{f(t)}{a} = \frac{f(ax+b)}{a}$$

NOTES

Example 4.5: Evaluate (i) $\int \tan x dx$ (ii) $\int \sec x dx$

Solution: (i) $\int \tan x dx = \int \frac{\sec x \tan x}{\sec x} dx = \log \sec x$

(ii) $\int \sec x dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx = \log (\sec x + \tan x)$

Example 4.6: Find $\int x\sqrt{x^2 + 1}\, dx$

Solution: We have,

$$\int x\sqrt{x^2 + 1}\, dx = \frac{1}{2}\int (2x)\,(x^2 + 1)^{\frac{1}{2}}\, dx$$

$$= \frac{1}{2}\frac{(x^2 + 1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$$

$$= \frac{1}{3}(x^2 + 1)^{3/2}$$

Example 4.7: Evaluate $\int \frac{x+1}{x^2 + 2x + 3}\, dx$

Solution: We have, $\int \frac{x+1}{x^2 + 2x + 3}\, dx = \frac{1}{2}\int \frac{2x+2}{x^2 + 2x + 3}\, dx$

$$= \frac{1}{2}\log (x^2 + 2x + 3)$$

Six Important Integrals

We will now evaluate the following six integrals:

(i) $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ (ii) $\int \frac{1}{\sqrt{a^2 + x^2}} dx$ (iii) $\int \frac{1}{\sqrt{x^2 - a^2}} dx$

(ii) $\int \sqrt{a^2 - x^2}\, dx$ (v) $\int \sqrt{a^2 + x^2}\, dx$ (vi) $\int \sqrt{x^2 - a^2}\, dx$

(i) To evaluate $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Put $x = a \sin \theta$, then, $dx = a \cos \theta\, d\theta$

Thus,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{a \cos \theta\, d\theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} = \int \frac{a \cos \theta}{a \cos \theta}\, d\theta$$

$$= \int 1.\, d\theta = \theta$$

$$= \sin^{-1}\left(\frac{x}{a}\right)$$

(ii) To evaluate $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx$

Put $x = a \sin h\, \theta$, then $dx = a \cos h\, \theta\, d\theta$

Thus,

$$\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \int \frac{a \cos h\, \theta\, d\theta}{\sqrt{a^2 + a^2 \sin h^2\, \theta}} = \int \frac{a \cos h\, \theta}{a \cos h\, \theta}\, d\theta$$

as $\cos h^2 \theta - \sin h^2 \theta = 1$

$$= \int d\theta = \theta = \sin h^{-1}\left(\frac{x}{a}\right)$$

NOTES

Aliter: Put $x = a \tan \theta$, then $dx = a \sec^2 \theta \, d\theta$

Thus,

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \int \frac{a \sec^2 \theta \, d\theta}{\sqrt{a^2 + a^2 \tan^2 \theta}} = \int \frac{\sec^2 \theta \, d\theta}{\sec \theta} = \int \sec \theta \, d\theta$$

$$= \log (\sec \theta + \tan \theta)$$

$$= \log \left[\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right]$$

$$= \log \left[\frac{x + \sqrt{x^2 + a^2}}{a}\right]$$

(iii) To evaluate $\int \frac{1}{\sqrt{x^2 - a^2}} dx$

Put $x = a \cos h \theta$, then $dx = a \sin h \theta \, d\theta$.

Thus,

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a \sin h \theta \, d\theta}{\sqrt{a^2 \cos h^2\theta - a^2}} = \int \frac{a \sin h \theta}{a \sin h \theta} d\theta = \int d\theta$$

$$= \theta = \cos h^{-1} \frac{x}{a}$$

Aliter: Put $x = a \sec \theta$, then $dx = a \sec \theta \tan \theta \, d\theta$.

Thus,

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a \sec \theta \tan \theta \, d\theta}{\sqrt{a^2 \sec^2 \theta - a^2}} = \int \sec \theta \, d\theta$$

$$= \log (\sec \theta + \tan \theta)$$

$$= \log \left[\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right]$$

$$= \log \left[\frac{x + \sqrt{x^2 - a^2}}{a}\right]$$

(iv) To evaluate $\int \sqrt{a^2 - x^2} \, dx$

Put $x = a \sin \theta$, then $dx = a \cos \theta \, d\theta$.

Thus,

$$\int \sqrt{a^2 - x^2} \, dx = \int \sqrt{a^2 - a^2 \sin^2 \theta} \, . \, a \cos \theta \, d\theta = \int a^2 \cos^2 \theta \, d\theta$$

$$= a^2 \int \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

NOTES

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$= \frac{a^2}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{a^2}{2}\left(\theta + \sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]$$

and hence,

$$\int\sqrt{a^2 - x^2}\;dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

(v) To evaluate $\int\sqrt{a^2 + x^2}\;dx$

Put $x = a \sin h\,\theta$, then $dx = a \cos h\,\theta\, d\theta$

Thus,

$$\int\sqrt{a^2 + x^2}\;dx = \int\sqrt{a^2 + a^2\sin h^2\theta}\;.\;a\cos h\,\theta\,d\theta$$

$$= \int a^2\cos h^2\theta\,d\theta$$

$$= a^2\int\frac{(\cos h\,2\theta + 1)}{2}\,d\theta \qquad (\text{As, } 2\cos h^2\theta = 1 + \cos h\,2\theta)$$

$$= \frac{a^2}{2}\left(\frac{\sin h\,2\theta}{2} + \theta\right)$$

$$= \frac{a^2}{2}[\sin h\,\theta\cos h\,\theta + \theta] \qquad (\text{As, } \sin h\,2\theta = 2\sin h\,\theta\cos h\,\theta)$$

$$= \frac{a^2}{2}\left[\sin h\,\theta\sqrt{1+\sin h^2\theta} + \theta\right]$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}} + \sin h^{-1}\frac{x}{a}\right]$$

and hence,

$$\int\sqrt{a^2 + x^2}\;dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sin h^{-1}\frac{x}{a}$$

Aliter: Put $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$

Thus,

$$\int\sqrt{a^2 + x^2}\;dx = \int\sqrt{a^2 + a^2\tan^2\theta}\;.\;a\sec^2\theta\,d\theta$$

$$= \int a^2\sec^3\theta\,d\theta$$

$$= \frac{a^2}{2}[\sec\theta\tan\theta + \log(\sec\theta + \tan\theta)]$$

$$= \frac{a^2}{2}\sqrt{1+\frac{x^2}{a^2}}\,.\,\frac{x}{a} + \frac{a^2}{2}\log\left[\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right]$$

$$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left[\frac{x+\sqrt{x^2+a^2}}{a}\right]$$

(vi) To evaluate $\int\sqrt{x^2-a^2}\,dx$

Put $x = a\cos h\,\theta$, then $dx = a\sin h\,\theta\,d\theta$

Thus,

$$\int\sqrt{x^2-a^2}\,dx = \int\sqrt{a^2\cos h^2\,\theta - a^2}\;a\sin h\,\theta\,d\theta$$

$$= a^2\int\sin h^2\,\theta\,d\theta$$

$$= a^2\int\frac{(\cos h\,2\theta-1)}{2}d\theta$$

$$= \frac{a^2}{2}\left(\frac{\sin h\,2\theta}{2}-\theta\right)$$

$$= \frac{a^2}{2}(\sin h\,\theta\cos h\,\theta - \theta)$$

$$= \frac{a^2}{2}\left[\sqrt{\cos h^2\theta-1}\,.\cos h\,\theta - \theta\right]$$

$$= \frac{a^2}{2}\left[\sqrt{\frac{x^2}{a^2}-1}.\frac{x}{a} - \cos h^{-1}x/a\right]$$

and hence,

$$\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cos h^{-1}x/a.$$

Aliter: Put $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta\,d\theta$

Thus

$$\int\sqrt{x^2-a^2}\,dx = \int\sqrt{a^2\sec^2\theta - a^2}\;a\sec\theta\tan\theta\,d\theta$$

$$= \int a^2\sec\theta\,.\tan^2\theta\,d\theta$$

$$= a^2\int\sec\theta\,(\sec^2\theta-1)\,d\theta$$

$$= a^2\int\sec^3\theta\,d\theta - a^2\int\sec\theta\,d\theta$$

$$= \frac{a^2}{2}[\sec\theta\tan\theta + \log(\sec\theta+\tan\theta)]$$

$$-a^2\log(\sec\theta+\tan\theta)\text{ [As in the previous case]}$$

$$= \frac{a^2}{2}\sec\theta\tan\theta - \frac{a^2}{2}\log(\sec\theta+\tan\theta)$$

$$= \frac{a^2}{2}\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1} - \frac{a^2}{2}\log\left[\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right]$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left[\frac{x+\sqrt{x^2-a^2}}{a}\right]$$

NOTES

Thus, we get following six results to remember:

NOTES

(i) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$

(ii) $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \sin h^{-1} \frac{x}{a}$

$= \log \left[\frac{x + \sqrt{x^2 + a^2}}{a} \right]$

(iii) $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cos h^{-1} \frac{x}{a}$

$= \log \left[\frac{x + \sqrt{x^2 - a^2}}{a} \right]$

(iv) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

(v) $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sin h^{-1} \frac{x}{a}$

$= \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log \left[\frac{x + \sqrt{a^2 + x^2}}{a} \right]$

(vi) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cos h^{-1} \frac{x}{a}$

$= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left[\frac{x + \sqrt{x^2 - a^2}}{a} \right]$

Example 4.8: Solve $\int \frac{1}{\sqrt{x^2 + x + 1}} dx$.

Solution: We have,

$$I = \int \frac{1}{\sqrt{x^2 + x + 1}} dx = \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} dx$$

Put $x + \frac{1}{2} = t$, then $dx = dt$

Thus,

$$I = \int \frac{dt}{\sqrt{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} = \sin h^{-1} \frac{t}{\sqrt{3}/2}$$

(By the second integral evaluated above)

$$= \sin h^{-1} \frac{x + 1/2}{\sqrt{3}/2} = \sin h^{-1} \frac{2x + 1}{\sqrt{3}}$$

The above result could, of course, be written directly without actually making the substitution $x + \frac{1}{2} = t$, by taking x as $x + \frac{1}{2}$ in the formula.

Methods of Substitution

NOTES

In this method, we express the given integral $\int f(x)\,dx$ in terms of another integral in which the independent variable x is changed to another variable t through some suitable relation $x = \phi(t)$.

Let $\qquad I = \int f(x)\,dx$

$$\frac{dI}{dx} = f(x)$$

$$\Rightarrow \qquad \frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = f(x)\,\frac{dx}{dt}$$

Thus, $\qquad I = \int f(x)\frac{dx}{dt}\,.\,dt = \int f[\varphi(t)]\,\varphi'(t)\,dt$

Note that we replace dx by $\varphi'(t)\,dt$, which we get from the relation $\frac{dx}{dt} = \varphi'(t)$ by *assuming* that dx and dt can be separated.

In fact, this is done only for convenience.

Example 4.9: Integrate $x(x^2 + 1)^3$.

Solution: Put $x^2 + 1 = t \Rightarrow 2x\frac{dx}{dt} = 1$

Thus, $\qquad \Rightarrow 2xdx = dt$

$$\int x(x^2+1)^3\,dx = \int \frac{1}{2}t^3\,dt = \frac{1}{2}\int t^3\,dt = \frac{1}{2}\frac{t^4}{4} = \frac{t^4}{8} = \frac{(x^2+1)^4}{8}$$

Example 4.10: Find $\int e^{\tan\theta}\sec^2\theta\,d\theta$.

Solution: Put $\tan\theta = t$, then $\sec^2\theta\,d\theta = dt$

Thus, $\int e^{\tan\theta}\sec^2\theta\,d\theta = \int e^t\,dt = e^t = e^{\tan\theta}$

4.3 DEFINITE INTEGRAL AND ITS PROPERTIES

Suppose $f(x)$ is a function such that

$$\int f(x)\,dx = g(x)$$

The definite integral $\int_a^b f(x)\,dx$ is defined by

$$\int_a^b f(x)\,dx = \{g(x)\}_a^b = g(b) - g(a)$$

where, a and b are two real numbers, and are called respectively, the *lower* and the *upper* limits of the integral.

Example 4.11: Evaluate $\int_0^{\frac{\pi}{2}} \cos x\,dx$

Solution: We know that $\int \cos x\,dx = \sin x$

Thus, $\int_0^{\frac{\pi}{2}} \cos x\, dx = \{\sin x\}_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0$

$= 1 - 0 = 1$

NOTES

Example 4.12: Find $\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2}\, dx$

Solution: Put $\tan^{-1} x = t$, then $\frac{1}{1+x^2}\, dx = dt$

Also, $x = \tan t$

Thus, when $x = 0$, $\tan t = 0 \Rightarrow t = 0$

When $x = 1$, $\tan t = 1 \Rightarrow t = \pi/4$

Hence,

$$\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2}\, dx = \int_0^{\pi/4} t^2\, dt = \left\{\frac{t^3}{3}\right\}_0^{\pi/4} = \frac{1}{3}\left(\frac{\pi}{4}\right)^3 - 0 = \frac{\pi^3}{192}$$

Note. In the above method, when we make the substitution, we also change the limits accordingly, the new limits being the values of the new variable which correspond to the values 0 and 1 of x. Alternatively, we could attempt the problem in the following way:

We first consider the integral $\int \frac{(\tan^{-1} x)^2}{1+x^2}\, dx$

i.e., we do not take limits. Then, as before, by the same substitution

$$\int \frac{(\tan^{-1} x)^2}{1+x^2}\, dx = \frac{t^3}{3} = \int \frac{(\tan^{-1} x)^3}{3}$$

and thus,

$$\int_1^0 \frac{(\tan^{-1} x)^2}{1+x^2}\, dx = \left\{\frac{(\tan^{-1} x)^3}{3}\right\}_0^1 = \frac{(\tan^{-1} 1)^3}{3} - \frac{(\tan^{-1} 0)^3}{3} = \frac{(\pi/4)^3}{3} - 0$$

$$= \frac{\pi^3}{192}$$

It might be remarked here that although both the methods are correct, the first method will prove very helpful in certain cases.

4.3.1 Properties of Definite Integrals

It is assumed that the function $f(x)$ is integrable on the closed interval (a, b),

1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
2. If $f_1(x)$, $f_2(x)$ are integrable on (a, b)

$$\int_a^b [f_1(x) + f_2(x)]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx$$

3. A definite integral can be expressed as a sum of definite integrals (additive property),

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \qquad a < c < b$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx, \quad a < c < d < b$$

It means that the area under the curve between (a, b) is the sum of the areas under the curve between (a, c), (c, d) and (d, b). This property is useful in finding the areas under some discontinuous functions also.

NOTES

4. A definite integral is a *function* only *of its limits a* and *b* and not of the variable which may be changed,

$$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz$$

5. A definite integral equals zero when the limits of integration are identical,

$$\int_a^a f(x)\,dx = [f(x)]_a^a = f(a) - f(a) = 0$$

The area on a single point is zero because the width *dx* of the rectangle, is zero.

6. The *directed length* of the interval of integration is given by,

$$\int_a^b dx = b - a$$

7. If the limits are interchanged, the sign of the definite integral changes,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

For example, $\int_2^4 x^2 dx = -\int_4^2 x^2 dx$

8. If one of the limits is the variable itself, the definite integral becomes equal to the indefinite integral of the function,

$$\int_a^x f(x)dx = f(x) - f(a) = f(x) + C$$

Where $C = -f(a)$ is a constant.

Note: When one or both of the limits are infinite we have *improper* integrals. The concept of limits is to be used in such cases to find the value of the definite integral.

For example, $\int_2^\infty \frac{dx}{x^2} = \underset{n\to\infty}{\text{Lim}} \int_2^n \frac{dx}{x^2} = \underset{n\to\infty}{\text{Lim}} \left[\frac{-1}{x}\right]_2^n = \underset{n\to\infty}{\text{Lim}} \left[-\frac{1}{n} + \frac{1}{2}\right] = \frac{1}{2}$

Other Useful Properties of Definite Integrals

9. $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

10. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)\,dx$

$$= 2\int_0^a f(x)dx, \text{ if } f(x) = f(2a - x)$$

$$= 0, \text{ if } f(x) = -f(2a - x)$$

11. $\int_a^b xf(x)dx = \frac{a+b}{2}\int_a^b f(x)dx$

12. $\int_0^a f(x)dx = 0 \text{ if } f(a - x) = f(a + x)$

For example, $I = \int_0^{\pi} \log(1+\cos\theta)\, d\theta = \int_0^{\pi} \log(1-\cos\theta)\, d\theta$

(Property 9. Note that $\cos(\pi - \theta) = -\cos\theta$)

NOTES

$$2I = \int_0^{\pi} \log(1-\cos^2\theta)\, d\theta = 2\int_0^{\pi} \log\sin\theta\, d\theta$$

(By adding, $\log(1+\cos\theta) + \log(1-\cos\theta) = \log(1-\cos^2\theta)$)

$$= 2\left[2\int_0^{\pi/2} \log\sin\theta\, d\theta\right] = 4\left(-\frac{\pi}{2}\log 2\right)$$

$\therefore \qquad I = -\pi \log 2$

4.4 CONCEPT OF INDEFINITE INTEGRAL

We will now learn a formula which will help us in finding the integral of a product of two functions.

We know that if u and v are two functions of x

Then, $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow \qquad u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides with respect to x, we get

$$\int u\frac{dv}{dx}\, dx = \int \frac{d}{dx}(uv)\, dx - \int v\frac{du}{dx}\, dx$$

or $$\int u\frac{dv}{dx}\, dx = uv - \int v.\frac{du}{dx}\, dx$$

Put $u = f(x)$, $\frac{dv}{dx} = g(x)$, then $v = \int g(x)dx$

The above reduces to

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [f'(x)\int g(x)\, dx]$$

where $f'(x)$ denotes the derivative of $f(x)$. This is the required formula. In words, integral of the product of two functions

= First function × Integral of the second – Integral of

(Differential of first × Integral of the second function).

It is clear from the formula that it is helpful only when we know (or can easily evaluate) integral of at least one of the two given functions. Here, one thumb rule may be followed by remembering a keyword 'ILATE'. I-means inverse function, L-means logarethmic, A-means algebraic, T-means trigonometric and E-means exponential. The following examples will illustrate how to apply this rule.

Example 4.13: Find $\int x^2 e^x dx$.

Solution: Taking x^2 as the first function as it is algebraic function and e^x as the second function since it is exponential. We note that,

$$\int x^2 e^x dx = x^2 e^x - \int (2x) e^x dx$$

$$= x^2 e^x - 2\int x\, e^x dx$$

Check Your Progress

1. What is constant of integration?
2. Prove that differentiation and integration cancel each other.
3. What are lower and upper limits of an integral?
4. What is the value of the definite integral when the upper limit is equal to the lower limit?
5. What is the directed length of the interval of integration?
6. How is the value of the definite integral affected if one of the limits is the variable itself?

$$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right] \qquad \text{(Integrating by parts again)}$$

$$= x^2e^x - 2[xe^x - e^x]$$

Note: If we had taken e^x as the first function and x^2 as the second function, we would not have got the answer.

NOTES

Example 4.14: Evaluate (i) $\int \log x\,dx$, (ii) $\int x^n \log x\,dx\ (n \neq -1)$

Solution: (i) We have, $\int \log x\,dx = \int 1.\log x\,dx$

$$= (\log x).\,x - \int x.\frac{1}{x}\,dx$$

$$= x\log x - x$$

Here, we have taken log x as first function and $1 = x^0$ as second function since it is algebraic.

(ii) We have $\int x^n \log x\,dx = (\log x)\dfrac{x^{n+1}}{n+1} - \int \dfrac{x^{n+1}}{n+1} \times \dfrac{1}{x}\,dx$

$$= \int \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n\,dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\cdot\frac{x^{n+1}}{n+1}$$

$$= \frac{x^{n+1}}{n+1}\left[\log x - \frac{1}{n+1}\right]$$

Here, we have taken log x as first function and $1 = x^0$ as second function since it is algebraic.

Example 4.15: Evaluate $\dfrac{x\,dx}{1+\cos x}$

Solution: We have,

$$\int \frac{x\,dx}{1+\cos x}\,dx = \int \frac{x}{2\cos^2\frac{x}{2}}\,dx = \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[2x\tan\frac{x}{2} - \int 2\tan\frac{x}{2}\,dx\right]$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} - \left(-2\log\cos\frac{x}{2}\right)$$

$$= x\tan\frac{x}{2} + 2\log\cos\frac{x}{2}$$

Here also, the thumb rule of 'ILATE' is applied. In the given two functions, one is algebraic and another is trigonometric. Algebraic function has been taken as first function and trigonometric as second function.

NOTES

4.4.1 How to Evaluate the Integrals

Consider the following examples to evaluate the integrals.

(i) $\int e^x [f(x) + f'(x)]\, dx$

(ii) $I_1 = \int e^{ax} \sin(bx + c)\, dx$

(iii) $I_2 = \int e^{ax} \cos(bx + c)\, dx$

The followings are the solutions for the above problems:

(i) Consider $\int e^x f(x)\, dx$

Integration by parts yields

$$\int e^x f(x)\, dx = f(x)e^x - \int f'(x)\, e^x\, dx$$

$$\Rightarrow \int e^x f(x)\, dx + \int e^x f'(x)\, dx = f(x)\, e^x$$

i.e., $\int e^x [f(x) + f'(x)]\, dx = f(x)\, e^x$

(ii) Using integration by parts, we find

$$I_1 = \int e^{ax} \sin(bx + c)\, dx$$

$$= \int \frac{e^{ax}}{a} \sin(bx + c) - \int \frac{e^{ax}}{a} . \cos(bx + c).\, b\, dx$$

$$= \int \frac{e^{ax}}{a} \sin(bx + c) - \frac{b}{a} I_2$$

Similarly,

$$I_2 = \int e^{ax} \cos(bx + c)\, dx$$

$$= \frac{e^{ax}}{a} \cos(bx + c) - \int -\frac{e^{ax}}{a} \sin(bx + c)\, bdx$$

$$= \int \frac{e^{ax}}{a} \cos(bx + c) + \frac{b}{a} I_1.$$

and thus,

$$I_1 = \frac{e^{ax}}{a} \sin(bx + c) - \frac{b}{a}\left[\frac{e^{ax}}{a}\cos(bx + c) + \frac{b}{a} I_1\right]$$

$$\Rightarrow \left(1 + \frac{b^2}{a^2}\right) I_1 = \frac{e^{ax}}{a} \sin(bx + c) - \frac{b}{a^2} e^{ax} \cos(bx + c)$$

$$\Rightarrow \left(\frac{a^2 + b^2}{a^2}\right) I_1 = e^{ax}\left[\frac{a \sin(bx + c) - b \cos(bx + c)}{a^2}\right]$$

$$\Rightarrow I_1 = \int e^{ax} \sin(bx + c)\, dx = e^{ax}\left[\frac{a \sin(bx + c) - b \cos(bx + c)}{a^2 + b^2}\right]$$

Similiarly,

$$I_2 = e^{ax}\left[\frac{a \cos(bx + c) + b \sin(bx + c)}{a^2 + b^2}\right]$$

NOTES

The above two integrals could be put into another form by the substitution

$$a = r\cos\theta,\ b = r\sin\theta$$

$$I_1 = e^{ax}\left[\frac{r\cos\theta.\sin(bx+c) - r\sin\theta\cos(bx+c)}{a^2+b^2}\right]$$

$$= e^{ax}\frac{r\sin(bx+c-\theta)}{a^2+b^2}$$

$$\Rightarrow \int e^{ax}\sin(bx+c)\,dx = e^{ax}\frac{\sin(bx+c-\tan^{-1}b/a)}{\sqrt{a^2+b^2}}$$ (As, $r^2 = a^2 + b^2$ and $\tan\theta = \frac{b}{a}$)

Similarly,

(iii) $$\int e^{ax}\cos(bx+c)\,dx = e^{ax}\frac{\cos(bx+c-\tan^{-1}b/a)}{\sqrt{a^2+b^2}}$$

Example 4.16: Find $\int e^x\,[\sin x + \cos x]\,dx$.

Solution: Since $\frac{d}{dx}(\sin x) = \cos x$

$$\int e^x(\sin x + \cos x)\,dx = e^x\sin x$$

Example 4.17: Find $\int \frac{xe^x}{(x+1)^2}dx$

Solution: We have $\int \frac{xe^x}{(x+1)^2}dx = \int e^x\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]dx$

$$= e^x.\frac{1}{x+1}$$ (As, $\frac{d}{dx}\left(\frac{1}{x+1}\right) = -\frac{1}{(x+1)^2}$)

Example 4.18: Evaluate $\int x^2\sin^{-1}x\,dx$.

Solution: We have, on integration by parts,

$$\int x^2\sin^{-1}x\,dx = (\sin^{-1}x)\frac{x^3}{3} - \int \frac{x^3}{3}.\frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int\frac{x^3}{\sqrt{1-x^2}}dx \quad \text{......................................} (1)$$

To evaluate $\int\frac{x^3}{\sqrt{1-x^2}}dx$, put $\sqrt{1-x^2} = t$

$$1 - x^2 = t^2$$

$\Rightarrow$ $-2x\,dx = 2t\,dt$

$\Rightarrow$ $x\,dx = -t\,dt$

Also, $x^2 = 1 - t^2$

Thus, $$\int\frac{x^3dx}{\sqrt{1-x^2}} = -\int\frac{(1-t^2)\,tdt}{t} = \int(t^2-1)\,dt = \frac{t^3}{3} - t$$

$$= \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2}$$

Hence, the required value is, [From Equation (1)]

NOTES

$$\int x^2 \sin^{-1} x dx = \frac{x^3}{3} \sin^{-1} x - \frac{1}{3}\left[\frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2}\right]$$

Integration of Algebraic Rational Functions

A function of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials in x, is called a *rational function.* We will now learn a few methods by which integration of such functions is done. We will be making an extensive use of partial fractions here.

Let us first establish the integrals

(1) $\int \frac{dx}{ax+b} = \frac{1}{a} \log (ax + b)$

(2) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

(3) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \frac{x-a}{x+a}$ $(x > a)$

(4) $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{a+x}{a-x}$ $(a > x)$

Assumed that $(a \neq 0)$.

Integrals (1) and (2) follow easily from the definition.

To evaluate integral (3), we note,

$$\frac{1}{x^2 - a^2} \equiv \frac{1}{(x-a)(x+a)} \equiv \frac{A}{x-a} + \frac{B}{x+a}$$

$\Rightarrow 1 \equiv A(x + a) + B(x - a)$

Putting, $x = -a$ and $x = a$, we get

$$A = \frac{1}{2a} \text{ and } B = -\frac{1}{2a}$$

Thus,

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

And hence,

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\int \frac{1}{x-a} dx - \frac{1}{2a}\int \frac{1}{x+a} dx$$

$$= \frac{1}{2a} \log (x-a) - \frac{1}{2a} \log (x+a)$$

$$= \frac{1}{2a} \log \frac{x-a}{x+a}$$

Integral (4) can be evaluated similarly, keeping in mind that,

$$\int \frac{1}{a-x} dx = -\log (a-x)$$

NOTES

Example 4.19: Integrate (i) $\frac{1}{4x^2 + 4x + 10}$

(ii) $\frac{1}{x^2 + x + 1}$

Solution: (i) We have $\int \frac{1}{4x^2 + 4x + 10} dx$

$$= \frac{1}{4}\int \frac{1}{x^2 + x + \frac{5}{2}} dx$$

$$= \frac{1}{4}\int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} dx$$

$$= \frac{1}{4} \cdot \frac{2}{3} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{3}{2}}\right) = \frac{1}{6} \tan^{-1} \left(\frac{2x + 1}{3}\right)$$

(ii) Also, $\int \frac{1}{x^2 + x + 1} dx = \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \frac{2}{\sqrt{3}} \cdot \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x + 1}{\sqrt{3}}\right)$$

Example 4.20: Evaluate $\int \frac{x^3}{x^2 + 8x + 12} dx$

Solution: We have,

$$\frac{x^3}{x^2 + 8x + 12} = x - 8 + \frac{52x + 96}{x^2 + 8x + 12} \quad \text{(By division)}$$

Again, $\frac{52x + 96}{x^2 + 8x + 12} = \frac{52x + 96}{(x + 2)(x + 6)}$

If $\frac{52x + 96}{(x + 2)(x + 6)} \equiv \frac{A}{x + 2} + \frac{B}{x + 6}$

then, $52x + 96 \equiv A(x + 6) + B(x + 2)$

Putting $x = -6$ and $x = -2$, we get

$A = -2, B = 54$

Hence,

$$\int \frac{x^3}{x^2 + 8x + 12} dx = \int \left(x - 8 + \frac{54x}{x + 6} - \frac{2}{x + 2} \right) dx$$

$$= \frac{x^2}{2} - 8x + 54 \log (x + 6) - 2 \log (x + 2)$$

4.4.2 Some More Methods

If the integrand consists of even powers of x only, then the substitution $x^2 = t$ is helpful while resolving into partial fractions.

NOTES

Note: The substitution is not to be made in the integral.

Example 4.21: Evaluate $\int \frac{x^2 dx}{(x^2+1)(3x^2+1)}$

Solution: Put $x^2 = t$ in $\frac{x^2}{(x^2+1)(3x^2+1)}$

Then, $$\frac{t}{(t+1)(3t+1)} \equiv \frac{A}{t+1} + \frac{B}{3t+1}$$

$\Rightarrow$ $$t \equiv A\,(3t+1) + B\,(t+1)$$

Putting $t = -1$ and $-\frac{1}{3}$, we get

$$A = \frac{1}{2},\ B = -\frac{1}{2}$$

$\Rightarrow$ $$\frac{t}{(t+1)(3t+1)} = \frac{1}{2(t+1)} - \frac{1}{2(3t+1)}$$

Thus,

$$\frac{x^2}{(x^2+1)(3x^2+1)} = \frac{1}{2(x^2+1)} - \frac{1}{2(3x^2+1)}$$

$$\Rightarrow \int \frac{x^2}{(x^2+1)(3x^2+1)}\,dx = \frac{1}{2}\int \frac{dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{3x^2+1}$$

$$= \frac{1}{2}\tan^{-1} x - \frac{1}{6}\int \frac{dx}{x^2+\frac{1}{3}}$$

$$= \frac{1}{2}\tan^{-1} x - \frac{1}{6}\cdot\frac{1}{1/\sqrt{3}}\tan^{-1}\frac{x}{1/\sqrt{3}}$$

$$= \frac{1}{2}\tan^{-1} x - \frac{x}{2\sqrt{3}}\tan^{-1}(x\sqrt{3})$$

Example 4.22: Solve $\int \frac{x^2+1}{x^4+1}\,dx$

Solution: We have, $$I = \int \frac{x^2+1}{x^4+1}\,dx = \int \frac{1+1/x^2}{x^2+1/x^2}\,dx$$

Put $$x - \frac{1}{x} = t$$

Then, $$\left(1+\frac{1}{x^2}\right) dx = dt$$

Also, $$x^2 + \frac{1}{x^2} - 2 = t^2$$

So, $I = \int \frac{dt}{t^2 + 2} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \qquad = \frac{1}{\sqrt{2}} \tan^{-1} \left[\frac{1}{\sqrt{2}} \left(x - \frac{1}{x} \right) \right]$

NOTES

Substitution before Resolving into Partial Fractions

The integration process is sometimes greatly simplified by a substitution as is seen in the following examples:

Example 4.23: Solve $\frac{dx}{x(x^4 - 1)}$

Solution: Put $x^4 = t$, then $4x^3\, dx = dt$

Thus, $\int \frac{dx}{x(x^4 - 1)} = \int \frac{x^3\, dx}{x^4 (x^4 - 1)} = \frac{1}{4} \int \frac{dt}{t(t - 1)}$

Now, $\frac{1}{t(t-1)} = \frac{1}{t-1} - \frac{1}{t}$

and hence, the given integral

$$= \frac{1}{4} \int \left(\frac{1}{t-1} - \frac{1}{t} \right) dt = \frac{1}{4} [\log (t-1) - \log t]$$

$$= \frac{1}{4} \log \left(\frac{t-1}{t} \right)$$

$$= \frac{1}{4} \log \left(\frac{x^4 - 1}{x^4} \right)$$

Example 4.24: Solve $\int_0^{\pi/4} \sqrt{\tan x}\, dx$

Solution: Put $\sqrt{\tan x} = t$, then $\tan x = t^2$

and $\sec^2 x\, dx = 2\, t\, dt$

$\Rightarrow \qquad dx = \frac{2t}{1 + \tan^2 x}\, dt = \frac{2t\, dt}{1 + t^4}$

Also, when $x = 0$, $t = \sqrt{\tan 0} = 0$

when $x = \frac{\pi}{4}$, $t = \sqrt{\tan \frac{\pi}{4}} = 1$

Hence, the given integral becomes

$$\int_0^1 \frac{t.2t\, dt}{1 + t^4} = 2 \int_0^1 \frac{t^2}{1 + t^4}\, dt$$

By integrating,

$$= 2 \left[\frac{\sqrt{2}}{8} \log \frac{t^2 - \sqrt{2}\, t + 1}{t^2 + \sqrt{2}\, t + 1} + \frac{1}{4} \sqrt{2} \tan^{-1} \frac{t^2 - 1}{t\sqrt{2}} \right]_0^1$$

$$= 2 \left[\left\{ \frac{\sqrt{2}}{8} \log \frac{2 - \sqrt{2}}{2 + \sqrt{2}} + \frac{\sqrt{2}}{4} \tan^{-1} 0 \right\} - \left\{ \frac{\sqrt{2}}{8} \log 1 - \frac{\sqrt{2}}{4} \tan^{-1} \infty \right\} \right]$$

$$= 2 \left[\frac{\sqrt{2}}{8} \log \frac{2 - \sqrt{2}}{2 + \sqrt{2}} + \frac{\sqrt{2}}{4} \frac{\pi}{2} \right]$$

$$= \frac{\sqrt{2}}{4} \log \frac{2-\sqrt{2}}{2+\sqrt{2}} + \frac{\sqrt{2}}{4}\pi = \frac{1}{2\sqrt{2}} \log \frac{2-\sqrt{2}}{2+\sqrt{2}} + \frac{1}{2\sqrt{2}}\pi$$

NOTES

Integrals of the Type $\int \frac{dx}{a + b\cos x + c\sin x}$, $b^2 + c^2 \neq 0$

The substitution $\tan \frac{x}{2} = t$, converts every rational function of sin x and cos x into a rational function of t and we can then evaluate the integral by using the previous methods.

Example 4.25: Evaluate (i) $\int_0^{\pi/2} \frac{dx}{4 + 5\cos x^2}$ *(ii)* $\int_0^{\pi} \frac{dx}{5 + 3\cos x}$

Solution: (i) Put $\tan \frac{x}{2} = t$, then $\frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt$

$$\Rightarrow \qquad dx = \frac{2\, dt}{1 + \tan^2 x/2} = \frac{2\, dt}{1 + t^2}$$

Also, as $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2} = \frac{1 - t^2}{1 + t^2}$

The given integral reduces to,

$$\int_0^1 \frac{2\,dt}{(1+t^2)\left[4 + \frac{5-5t^2}{1+t^2}\right]} \qquad \begin{bmatrix} \text{Note, when} & x = 0 \\ & t = \tan 0 = 0 \\ \text{When} & x = \pi/2 \\ & t = \tan \pi/4 = 1 \end{bmatrix}$$

$$= 2\int_0^1 \frac{dt}{4 + 4t^2 + 5 - 5t^2} = 2\int_0^1 \frac{dt}{9 - t^2}$$

$$= 2.\frac{1}{2\times 3}\left[\log \frac{3+t}{3-t}\right]_0^1$$

$$= \frac{1}{3}\log \frac{3+1}{3-1} - \frac{1}{3}\log \frac{3}{3}$$

$$= \frac{1}{3}\log 2$$

(ii) Put $\tan \frac{x}{2} = t$, then $\frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt$

$$\Rightarrow \qquad dx = \frac{2\, dt}{1 + \tan^2 x/2} = \frac{2dt}{1 + t^2}$$

Also as $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2} = \frac{1 - t^2}{1 + t^2}$

The given integral reduces to,

$$\int_0^{\infty} \frac{2\,dt}{(1+t^2)\left[5 + 3\frac{(1-t^2)}{1+t^2}\right]} \qquad \begin{bmatrix} \text{Note, when } x = 0,\ t = \tan 0 = 0 \\ \text{When } x = \pi,\ t = \tan \frac{\pi}{2} = \infty \end{bmatrix}$$

NOTES

$$= 2\int_0^{\infty} \frac{dt}{5+5t^2+3-3t^2} = 2\int_0^{\infty} \frac{dt}{2t^2+8} = \int_0^{\infty} \frac{dt}{t^2+4}$$

$$= \left\{\frac{1}{2}\tan^{-1}\frac{t}{2}\right\}_0^{\infty} = \frac{1}{2}\tan^{-1}\infty - \frac{1}{2}\tan^{-1}0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Integration of $\dfrac{a\cos x + b\sin x}{c\cos x + d\sin x}$, $(a^2+b^2)(c^2+d^2) \neq 0$

We determine two constants λ and μ such that,

$a\cos x + b\sin x = \lambda(-c\sin x + d\cos x) + \mu(c\cos x + d\sin x)$

where $-c\sin x + d\cos x = \dfrac{d}{dx}(c\cos x + d\sin x)$

Comparing coefficients of $\cos x$ and $\sin x$, we get

$$a = \lambda d + \mu c$$
$$b = -\lambda c + \mu d$$

$\Rightarrow$ $$\lambda = \frac{ad-bc}{d^2+c^2},\ \mu = \frac{ac-bd}{d^2+c^2}$$

Hence,

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\,dx = \lambda\int\frac{-c\sin x + d\cos x}{c\cos x + d\sin x}\,dx + \mu\int 1.dx$$

$$= \lambda\log(c\cos x + d\sin x) + \mu x$$

Integration of $\dfrac{a\cos x + b\sin x + c}{d\cos x + e\sin x + f}$

In this case, we determine three constants λ, μ, ν, such that $a\cos x + b\sin x + c = \lambda(d\cos x + e\sin x + f) + \mu(-d\sin x + e\cos x) + \nu$ and proceed as in the earlier case.

Example 4.26: Find $\displaystyle\int \frac{4\sin x + 2\cos x + 3}{2\sin x + \cos x + 3}\,dx$

Solution: We determine λ, μ, ν such that,

$4\sin x + 2\cos x + 3 = \lambda(2\sin x + \cos x + 3) + \mu(2\cos x - \sin x) + \nu$

Comparing coefficients of $\sin x$, $\cos x$ and the constant terms, we get

$$4 = 2\lambda - \mu,$$
$$2 = \lambda + 2\mu$$
$$3 = 3\lambda + \nu$$

$\Rightarrow$ $$\lambda = 2, \mu = 0, \nu = -3$$

Thus,

$4\sin x + 2\cos x + 3 = 2(2\sin x + \cos x + 3) + 0\,(2\cos x - 1) - 3$

$$\Rightarrow \int\frac{4\sin x + 2\cos x + 3}{2\sin x + \cos x + 3}\,dx = 2\int 1.\,dx - 3\int\frac{1}{2\sin x + \cos x + 3}\,dx$$

$$= 2x - 3\int\frac{1}{2\sin x + \cos x + 3}\,dx$$

NOTES

Now, solve $\int \frac{1}{2\sin x + \cos x + 3}\, dx$

We put $\tan\frac{x}{2} = t$, then, $dx = \frac{2\,dt}{1+t^2}$ and the integral,

$$= \int \frac{2dt}{(1+t^2)\left[\frac{2.2t}{t^2+1} + \frac{1-t^2}{1+t^2} + 3\right]}$$

$$= \int \frac{2\,dt}{4t + 1 - t^2 + 3 + 3t^2}$$

$$= \int \frac{dt}{t^2 + 2t + 2}$$

$$= \int \frac{dt}{(t+1)^2 + 1}$$

$$= \tan^{-1}(t+1) = \tan^{-1}\left(1 + \tan\frac{x}{2}\right)$$

Hence, the required result is,

$$2x - 3\tan^{-1}\left(1 + \tan\frac{x}{2}\right)$$

Integration of Irrational Functions

Consider the types $\int \left(\frac{a_1x + a_2}{b_1x + b_2}\right)^{1/n} dx$ and $\int \frac{dx}{(a+bx)^n}$

These may be found by the substitution $u^n = \frac{a_1x + a_2}{b_1x + b_2}$ and $u = (a + bx)$.

For example, let us Integrate, $\int \sqrt{ax^2 + bx + c}\; dx \quad (a < 0, b^2 - 4ac > 0)$

$$= \int (\sqrt{-a}) \sqrt{\frac{b^2 - 4ac}{4a^2} - \left(x + \frac{b}{2a}\right)^2}\, dx$$

$$= \sqrt{-a}\left[\frac{x + b/2a}{2}\sqrt{\frac{b^2-4ac}{4a^2} - \left(x + \frac{b}{2a}\right)^2} + \frac{b^2 - 4ac}{8a^2}\sin^{-1}\frac{x + b/2a}{\sqrt{(b^2-4ac)/4a^2}} + C\right]$$

Similarly $\int (ax^2 + bx + c)^{-1/2} dx = \frac{1}{\sqrt{-a}} \sin^{-1}\frac{x + b/2a}{\sqrt{(b^2 - 4ac)/4a^2}} + C$

If, in this case, the numerator is a linear function of x, it can be broken into two parts.

Example 4.27: Integrate $\int \frac{dx}{x+\sqrt{x^2+x+2}}$

Solution: Put, $u = \sqrt{x^2+x+2} + x$ or $x = \frac{u^2-2}{1+2u}$

NOTES

$$dx = \frac{2(u^2+u+2)}{(1+2u)^2} \text{ and } \sqrt{x^2+x+2} = \frac{u^2+u+2}{1+2u}$$

$$\therefore \quad \int \frac{dx}{x+\sqrt{x^2+x+2}} = 2\int \frac{(u^2+u+2)du}{u(1+2u)^2}$$

Now, $\frac{u^2+u+2}{u(1+2u)^2} = \frac{A}{u} + \frac{B}{1+2u} + \frac{C}{(1+2u)^2}$

$\Rightarrow \quad u^2 + u + 2 = A(1+2u)^2 + Bu(1+2u) + cu$

Putting $u = 0$, $\frac{-1}{2}$, we get $A = 2$, $C = \frac{-7}{2}$

Again, comparing coefficients of x^2 on both sides,

we get, $1 = 4A + 2B \Rightarrow B = \frac{-7}{2}$

Hence,

$$\int \frac{dx}{x+\sqrt{x^2+x+2}} = 2\int \left(\frac{2}{u} - \frac{7}{2(1+2u)} - \frac{7}{2(1+2u)^2} \right) du$$

$$= 4\log u - \frac{7}{2}\log(1+2u) + \frac{7}{2(1+2u)} + C$$

where, $u = \sqrt{x^2+x+2} + x$

4.5 INTEGRAL AS ANTIDERIVATIVE

In calculus, an **antiderivative**, **primitive integral** or **indefinite integral** of a function *f* is a differentiable function *F* whose derivative is equal to the original function *f*. This can be stated symbolically as $F' = f$. The process of solving for antiderivatives is called **antidifferentiation** (or **indefinite integration**) and its opposite operation is called differentiation, which is the process of finding a derivative.

Antiderivatives are related to definite integrals through the fundamental theorem of calculus: the definite integral of a function over an interval is equal to the difference between the values of an antiderivative evaluated at the endpoints of the interval. The discrete equivalent of the notion of antiderivative is antidifference.

Uses and Properties

Antiderivatives are important because they can be used to compute definite integrals, using the fundamental theorem of calculus: if *F* is an antiderivative of the integrable function *f* and *f* is continuous over the interval $[a, b]$, then:

Check Your Progress

7. What is the integral of the product of two functions?
8. Which thumb rule is followed for finding the integral of the product of two functions?
9. What is a rational function?
10. Which substitution is done if the integrand consists of even powers of *x* only?

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

NOTES

Because of this, each of the infinitely many antiderivatives of a given function *f* is sometimes called the "general integral" or "indefinite integral" of *f* and is written using the integral symbol with no bounds:

$$\int f(x)\mathrm{d}x.$$

If *F* is an antiderivative of *f*, and the function *f* is defined on some interval, then every other antiderivative *G* of *f* differs from *F* by a constant: there exists a number *C* such that $G(x) = F(x) + C$ for all *x*. *C* is called the arbitrary constant of integration. If the domain of *F* is a disjoint union of two or more intervals, then a different constant of integration may be chosen for each of the intervals. For instance

$$F(x) = \begin{cases} -\frac{1}{x} + C_1 & x < 0 \\ -\frac{1}{x} + C_2 & x > 0 \end{cases}$$

is the most general antiderivative of $f(x) = 1/x^2$ on its natural domain $(-\infty, 0) \cup (0, \infty)$.

Every continuous function *f* has an antiderivative, and one antiderivative *F* is given by the definite integral of *f* with variable upper boundary:

$$F(x) = \int_0^x f(t)\mathrm{d}t.$$

Varying the lower boundary produces other antiderivatives (but not necessarily all possible antiderivatives). This is another formulation of the fundamental theorem of calculus.

There are many functions whose antiderivatives, even though they exist, cannot be expressed in terms of elementary functions (like polynomials, exponential functions, logarithms, trigonometric functions, inverse trigonometric functions and their combinations). Examples of these are

$$\int e^{-x^2}\mathrm{d}x, \quad \int \sin x^2 \mathrm{d}x, \quad \int \frac{\sin x}{x}\mathrm{d}x, \quad \int \frac{1}{1\mathrm{n}x}\mathrm{d}x, \quad \int x^x \mathrm{d}x.$$

From left to right, the first four are the error function, the Fresnel function, the trigonometric integral, and the logarithmic integral function.

4.6 BETA AND GAMMA FUNCTIONS

In mathematics, the beta function, also called the Euler integral of the first kind, is a special function defined by

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

for $\mathrm{Re}(x), \mathrm{Re}(y) > 0.$

The beta function was studied by Euler and Legendre and was given its name by Jacques Binet; its symbol Β is a Greek capital β rather than the similar Latin capital B.

Properties

NOTES

The beta function is symmetric, meaning that

$$B(x, y) = B(y, x).$$

When x and y are positive integers, it follows from the definition of the gamma function Γ that:

$$B(x, y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

It has many other forms, including:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$B(x, y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} d\theta, \qquad \text{Re}(x) > 0,\ \text{Re}(y) > 0$$

$$B(x, y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} d\theta, \qquad \text{Re}(x) > 0,\ \text{Re}(y) > 0$$

$$B(x, y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt, \quad \text{Re}(x) > 0,\ \text{Re}(y) > 0$$

$$B(x, y) = \sum_{n=0}^{\infty} \frac{\binom{n-y}{n}}{x+n}.$$

$$B(x, y) = \sum_{n=0}^{\infty} \frac{\binom{n-y}{n}}{x+n}.$$

$$B(x, y) = \frac{x+y}{xy} \prod_{n=1}^{\infty} \left(1 + \frac{xy}{n(x+y+n)}\right)^{-1}$$

The Beta function satisfies several interesting identities, including

$$B(x, y) = B(x, y+1) + B(x+1, y)$$

$$B(x+1, y) = B(x, y) \cdot \frac{x}{x+y}$$

$$B(x, y+1) = B(x, y) \cdot \frac{y}{x+y}$$

$$B(x, y) \cdot (t \to t_+^{x+y-1}) = (t \to t_+^{x-1}) \times (t \to t_+^{y-1}) \quad x \geq 1,\ y \geq 1$$

$$B(x, y) \cdot B(x+y, 1-y) = \frac{\pi}{x \sin(\pi y)}$$

where $t \to t_+^x$ is a truncated power function and the star denotes convolution. The lowermost identity above shows in particular $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Some of these identities, e.g. the trigonometric formula, can be applied to deriving the volume of an n-ball in Cartesian coordinates.

NOTES

Euler's integral for the beta function may be converted into an integral over the Pochhammer contour *C* as

$$(1-e^{2\pi i\alpha})(1-e^{2\pi i\beta})\mathrm{B}(\alpha,\beta) = \int_C t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t.$$

This Pochhammer contour integral converges for all values of α and β and so gives the analytic continuation of the beta function.

Just as the gamma function for integers describes factorials, the beta function can define a binomial coefficient after adjusting indices:

$$\binom{n}{k} = \frac{1}{(n+1)\mathrm{B}(n-k+1, k+1)}$$

Moreover, for integer *n*, B can be factored to give a closed form, an interpolation function for continuous values of *k*:

$$\binom{n}{k} = (-1)^n n! \frac{\sin(\pi k)}{\pi \prod_{i=0}^{n}(k-i)}.$$

The beta function was the first known scattering amplitude in string theory, first conjectured by Gabriele Veneziano. It also occurs in the theory of the preferential attachment process, a type of stochastic urn process.

Gamma Function

In mathematics, the **gamma function** (represented by the capital Greek letter **Γ**) is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers. That is, if *n* is a positive integer:

$$\Gamma(n) = (n-1)!.$$

The gamma function is defined for all complex numbers except the non-positive integers. For complex numbers with a positive real part, it is defined via a convergent improper integral:

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx.$$

This integral function is extended by analytic continuation to all complex numbers except the non-positive integers (where the function has simple poles), yielding the meromorphic function we call the gamma function. In fact the gamma function corresponds to the Mellin transform of the negative exponential function:

$$\Gamma(t) = \{Me^{-x}\}(t).$$

The gamma function is a component in various probability-distribution functions, and as such it is applicable in the fields of probability and statistics, as well as combinatorics.

NOTES

The notation $\Gamma(t)$ is due to Legendre. If the real part of the complex number t is positive ($\mathrm{Re}(t) > 0$), then the integral

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx.$$

converges absolutely, and is known as the **Euler integral of the second kind** (the Euler integral of the first kind defines the Beta function). Using integration by parts, we see that the gamma function satisfies the functional equation:

$$\Gamma(t+1) = t\Gamma(t).$$

Combining this with $\Gamma(1) = 1$, we get:

$$\Gamma(n) = 1\cdot 2\cdot 3\cdots (n-1) = (n-1)!$$

for all positive integers n.

The identity $\Gamma(t) = \Gamma(t+1)/t$ can be used (or, yielding the same result, analytic continuation can be used) to extend the integral formulation for $\Gamma(t)$ to a meromorphic function defined for all complex numbers t, except t = ”n for integers n e” 0, where the function has simple poles with residue (“1)^n/n!.

It is this extended version that is commonly referred to as the gamma function.

Relationship between Gamma Function and Beta Function

To derive the integral representation of the beta function, write the product of two factorials as

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-u}u^{x-1}du \int_0^\infty e^{-\upsilon}\upsilon^{y-1}d\upsilon$$

$$= \int_0^\infty \int_0^\infty e^{-u-\upsilon}u^{x-1}\upsilon^{y-1}\mathrm{d}u\mathrm{d}\upsilon.$$

Changing variables by $u = f(z, t) = zt$ and $\upsilon = g(z,t) = z(1-t)$ shows that this is

$$\Gamma(x)\Gamma(y) = \int_{\approx=0}^\infty \int_{t=0}^1 e^{-z}(zt)^{x-1}(z(1-t))^{y-1}|J(z,t)|\mathrm{d}t\mathrm{d}z$$

$$= \int_{\approx=0}^\infty \int_{t=0}^1 e^{-z}(zt)^{x-1}(z(1-t))^{y-1}z\mathrm{d}t\mathrm{d}z$$

$$= \int_{\approx=0}^\infty e^{-z}z^{x+y-1}\mathrm{d}z\int_{t=0}^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,$$

where $|J(z,t)|$ is the absolute value of the Jacobian determinant of $u = f(z,t)$ and $v = g(z,t)$.

Hence

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)\mathrm{B}(x,y).$$

The stated identity may be seen as a particular case of the identity for the integral of a convolution. Taking

$f(u) := e^{-u}u^{x-1}1_{\mathbb{R}+}$ and $g(u) := e^{-u}u^{y-1}1_{\mathbb{R}+}$, one has:

$$\Gamma(x)\Gamma(y) = \left(\int_{\mathbb{R}} f(u)\mathrm{d}u\right)\left(\int_{\mathbb{R}} g(u)\mathrm{d}u\right) = \int_{\mathbb{R}}(f \times g)(u)\mathrm{d}u = \mathrm{B}(x,y)\Gamma(x+y).$$

4.7 IMPROPER INTEGRAL

NOTES

In calculus, an **improper integral** is the limit of a definite integral as an endpoint of the interval(s) of integration approaches either a specified real number ∞ or $-\infty$ or, in some cases, as both endpoints approach limits. Such an integral is often written symbolically just like a standard definite integral, perhaps with *infinity* as a limit of integration.

Specifically, an improper integral is a limit of the form

$$\lim_{b\to\alpha}\int_a^b f(x)\mathrm{d}x, \qquad \lim_{a\to-\infty}\int_a^b f(x)\mathrm{d}x,$$

or of the form

$$\lim_{c\to b^-}\int_a^c f(x)\mathrm{d}x, \qquad \lim_{c\to a^+}\int_c^b f(x)\mathrm{d}x.$$

in which one takes a limit in one or the other (or sometimes both) endpoints (Apostol 1967, §10.23). When a function is undefined at finitely many interior points of an interval, the improper integral over the interval is defined as the sum of the improper integrals over the intervals between these points.

By abuse of notation, improper integrals are often written symbolically just like standard definite integrals, perhaps with *infinity* among the limits of integration. When the definite integral exists (in the sense of either the Riemann integral or the more advanced Lebesgue integral), this ambiguity is resolved as both the proper and improper integral will coincide in value.

Often one is able to compute values for improper integrals, even when the function is not integrable in the conventional sense (as a Riemann integral, for instance) because of a singularity in the function, or poor behavior at infinity. Such integrals are often termed "properly improper", as they cannot be computed as a proper integral.

Types of Integrals

There is more than one theory of integration. From the point of view of calculus, the Riemann integral theory is usually assumed as the default theory. In using improper integrals, it can matter which integration theory is in play.

- For the Riemann integral (or the Darboux integral, which is equivalent to it), improper integration is necessary *both* for unbounded intervals (since one cannot divide the interval into finitely many subintervals of finite length) *and* for unbounded functions with finite integral (since, supposing it is unbounded above, then the upper integral will be infinite, but the lower integral will be finite).
- The Lebesgue integral deals differently with unbounded domains and unbounded functions, so that often an integral which only exists as an improper Riemann integral will exist as a (proper) Lebesgue integral, such as $\int_1^\infty \frac{1}{x^2}\mathrm{d}x.$ On the other hand, there are also integrals that have an improper Riemann integral but do not have a (proper) Lebesgue integral, such as $\int_0^\infty \frac{\sin x}{x}\mathrm{d}x.$ The Lebesgue theory does not see this as a deficiency: from the point of view of measure

NOTES

theory, $\int_0^{\infty} \frac{\sin x}{x} dx = \infty - \alpha$ and cannot be defined satisfactorily. In some situations, however, it may be convenient to employ improper Lebesgue integrals as is the case, for instance, when defining the Cauchy principal value. The Lebesgue integral is more or less essential in the theoretical treatment of the Fourier transform, with pervasive use of integrals over the whole real line.

- For the Henstock–Kurzweil integral, improper integration *is not necessary*, and this is seen as a strength of the theory: it encompasses all Lebesgue integrable and improper Riemann integrable functions.

Improper Integral of $\int_1^{\infty} e^{-x^2} dx$

Study the convergence of $\int_1^{\infty} e^{-x^2 dx}$

We cannot evaluate the integral directly, e^{-x^2} does not have an antiderivative. We note that

$$x \geq 1 \Leftrightarrow x^2 \geq x$$
$$\Leftrightarrow -x^2 \leq -x$$
$$\Leftrightarrow e^{-x^2} \leq e^{-x}$$

Now,

$$\int_{}^{\infty} e^{-x} dx = \lim_{t\to\infty} \int_1^{t} e^{-x} dx$$
$$= \lim_{t\to\infty} (e^{-1} - e^{-t})$$
$$= c^{-1}$$

and therefore converges. It follows that $\int_1^{\infty} e^{-x^2}$ converges by the comparison theorem.

4.8 APPLICATIONS OF INTEGRAL CALCULUS (LENGTH, AREA, VOLUME)

Let $y = f(x)$ be a continuous function shown as a curve (refer Figure 4.1). To find the area under this curve in the interval (a, b) take a small strip of width $x_2 - x_1 = \Delta x_1$, its height being $f(x_1)$. The area of this strip is $f(x_1)$, Δx_1. If we similarly take n strips of width Δx_i $(i = 1, 2, ..., n)$ and n being the corresponding heights $f(x_i)$ $(i = 1, 2, ... ,n)$ we have n thin rectangles, each of area,

$$f(x_i)\ \Delta x_i,\ i = 1, 2, ..., n$$

The total of all these area is given by,

$$\sum_{i=1}^{n} f(x)\Delta x_{\iota}$$

NOTES

This is not exactly the area under the curve, but it can be, if the widths of the rectangles are taken sufficiently small, i.e., if n is very large or as n tends to infinity. The rectangles will become thinner (almost lines) and we can write the area between (a, b) under the curve $y = f(x)$ as the limit,

$$A = \operatorname*{Lim}_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

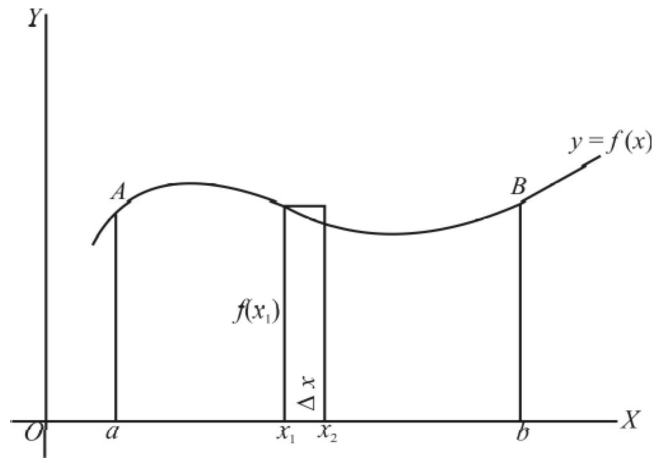


Fig. 4.1 *Continuous Function y = f(x)*

The area under a curve is thus expressed as a *discrete* sum.

In the limit we can write the area in the *continuous* form,

$$A = \int_a^b f(x)dx$$

The similarities between the expressions $\operatorname*{Lim}_{n\to\infty} \Sigma f(x_i)\Delta x_i$ and $\int_a^b f(x)\, dx$ may be noted.

The discrete quantitites $f(x_i)$ and Δx_i, have their continuous counterparts $f(x)$ and dx and the discrete summation sign Σ is replaced by the continuous summation sign $\int$. The area under the curve $y = f(x)$ between the limits a, b can thus be written as a definite integral,

$$\text{Area } abBA = \int_a^b f(x)dx = F(b) - F(a)$$

Example 4.28: Evaluate a definite integral as the limit of a sum by proving,

$$\int_a^b e^x dx = [e^x]_a^b = e^b - e^a$$

Solution: The procedure from the first principle is applied to get the limit of the sum. Let the interval (a, b) be divided into n subintervals each of size h at points

$a, a + h, a + 2h, \ldots, a + nh$, where $h = \dfrac{b-a}{n} \to 0$ as $n \to 0$.

NOTES

Let, $t_r = a + rh,\ f(t_r) = f(a + rh) = e^{a+rh} = e^a.e^{rh}$

$$S_n = \sum_{r=1}^{n} f(t_t)\,\Delta_r = \sum_{r=1}^{n}(e^a.e^{rh})h = e^a h \sum_{r=1}^{n} e^{rh}$$

$$= e^a h\left[e^h + e^{2h} + \ldots.. + e^{nh}\right]$$

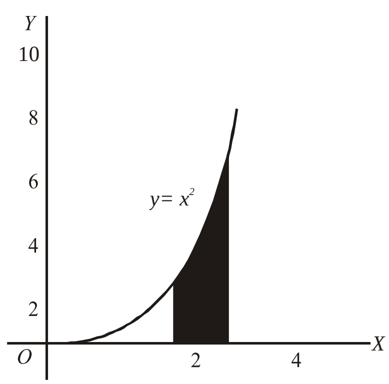
$$= e^a.h.e^h \frac{(e^{nh}-1)}{(e^h-1)} = \frac{he^h}{e^h-1}(e^{b-a}-1) \qquad (\because\ b-a = nh)$$

$$= (e^b - e^a)e^h.\frac{h}{e^h-1}$$

As $\begin{matrix} h \to 0 \\ n \to \infty \end{matrix}$ $S_n = (e^b - e^a)(e^h)\left(\dfrac{h}{e^h-1}\right) \to e^b - e^a$

Since, $\lim\limits_{n\to 0} e^h = 1,\ \lim\limits_{h\to 0}\dfrac{h}{e^h-1} = 1$

Example 4.29: Find the area bounded by the x-axis and the curve $y = x^2$ between $x = 1$ and $x = 3$.



Solution: $\displaystyle\int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1^3}{8} = 8\frac{2}{3}$

Example 4.30: Find the area under the curve,

$y = \sqrt{x} \qquad 1 \le x \le 4$

Solution: $\displaystyle\int_1^4 \sqrt{x}dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}\left[4^{3/2} - 1^{3/2}\right] = \frac{2}{3}\times 7 = \frac{14}{3}$

Sign Convention

NOTES

If the function $y = f(x)$ is positive in the interval (a, b) and the curve is above the x-axis then $\int_a^b f(x)dx$ is positive.

If $y = f(x)$ is negative in the interval (a, b) and the curve is below x-axis then, $\int_a^b f(x)dx$ is negative.

If $y = f(x)$ changes sign in the interval and the curve crosses the x-axis, the area is the algebraic sum of a positive area and a negative area.

For example, to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, consider the ellipse divided into 4 equal parts (refer Figure 4.2). The area of part Oab is,

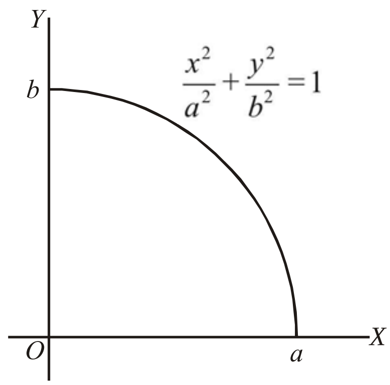


Fig. 4.2 *Part Oab of Ellipse*

$$\int_0^a y dx = \int_0^a \frac{b}{a} \sqrt{a^2 - x^2}\, dx$$

$$= \frac{b}{a} \frac{\pi a^2}{4} = \frac{\pi ab}{4}$$

$\therefore$ Area of the ellipse $= \pi\, ab$.

We can also prove that the area between 0 and 2 of the curve $4y = x^3 + x – 2$ is the sum of two areas (refer Figure 4.3).

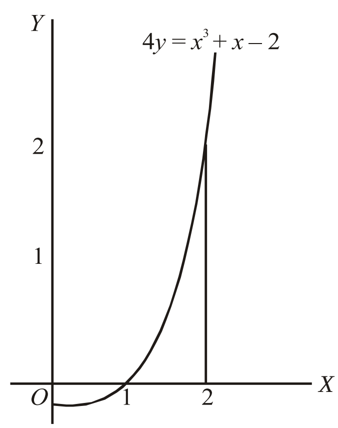


Fig. 4.3 *Curve* $4y = x^2 + x - 2$

$$\int_0^1 | f(x) | dx + \int_1^2 | f(x) | dx = \frac{5}{16} + \frac{13}{16} = \frac{9}{8}$$

Here the algebraic sum of a positive area and a negative area is taken.

NOTES

Limits of Integration Infinity

If $f(x)$ is continuous over $a \le x \le b$, we define $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$ provided the limit exists.

Similarly, $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$

and, $\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty} \int_a^{a1} f(x)\, dx + \lim_{b\to\infty} \int_{a1}^b f(x)\, dx$

If $f(x)$ has one or more points of discontinuity over $a \le x \le b$, or at least one of the limits of integration is ∞ as in the above cases, we have an improper integral.

For example, $\int_0^\infty e^{-x} \sin x dx = \left[\frac{-e^{-x}}{2} (\sin x + \cos x) \right]_0^\infty$

This may be evaluated by writing,

$$\lim_{b\to\infty} \frac{-e^{-x}}{2} [\sin x + \cos x]_0^b = -\frac{1}{2}(0 - 1) = \frac{1}{2}$$

Note: $e^{-\infty} = 0$, $\sin \infty$ or $\cos \infty$ lie between ± 1 and $e^0 = 1$.

4.9 MULTIPLE INTEGRALS

4.9.1 The Double Integrals

Let $f(x, y)$ be a function of the two real variables defined at every point (x, y) in the region R of the (x, y) plane, bounded by a closed curve C. Let the region R be subdivided in any manner, into n subregions (denoted as) $\Delta A, \Delta A_2, \Delta A_3, ..., \Delta A_n$. Let (x_r, y_r) be any point in the subregion ΔA_r. Let S denote the sum,

$$\text{i.e., } S = \sum_{r=1}^{n} f(x_r, y_r)\Delta A_r$$

If the limit of the sum S exists, as $n \to \infty$ and as each sub region $\Delta A_r \to 0$, and the limit is independent of the manner in which the region R is subdivided and the points (x_r, y_r) chosen in the region A_r, then that limit is called the double integral of $f(x, y)$ over the region R. It is denoted as $\int_R \int f(x_r y_r) dA$

Thus, $\iint_R f(x_r, y_r) dA = LT_{n\to\infty} \sum_1^n f(x_r, y_r)\Delta A_r$

The double integral $\iint_R f(x_r, y_r) dA$ is often denoted as $\iint_R f(x, y) dxdy$.

NOTES

It can be shown that, if $f(x, y)$ is a continuous function of x and y in the region R, then the limit of S exists independent of the mode of subdivision of R and points chosen in the sub-regions.

The following are some properties of the double integral.

(i) $\iint_R f(x, y) + g(x, y)dA = \iint_R f(x, y)dA + \iint_R g(x, y)dA$

(ii) $\iint_R kf(x, y)dA = k\iint_R f(x, y)dA$ for any constant k.

(iii) $\iint_R f(x, y)dA = \iint_{R_1} f(x, y)dA + \iint_{R_2} f(x, y)dA$

If the region R is composed of the two disjoint regions R_1 and R_2.

4.9.2 Evaluation of Double Integrals in Cartesian and Polar Coordinates

The evaluation of certain double integrals becomes easier by effecting a change in the variables. Sometimes when the change is effected from Cartesian coordinates to polar coordinates, the integral reduces to a simpler form. For example, when the integral is a function of $x^2 + y^2$. the transformation $x = r\cos\theta$, $y = r\sin\theta$ makes it function of r^2.

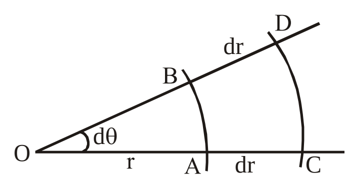


Fig. 4.4 *Segment ACDBA*

To transform to polar coordinates we put $x = r\cos\theta$, $y = r\sin\theta$.

Regarding the elements of area dA, considering the area $ACDBA$ as a rectangle (refer Figure 4.4),

$$dA = \text{arc } AB.\, AC$$
$$= r\,d\theta.\, dr = r\,dr\,d\theta$$

Hence, $\iint_R (x, y)dx\,dy = \iint_R f(r\cos\theta,\ r\sin\theta)r\,dr\,d\theta$

Where R is the region of integration.

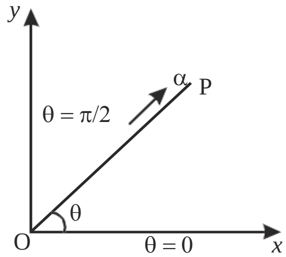
Note: The boundaries of the region R will to be expressed in polar coordinates to facilitate the fixing of limits of integration.

Example 4.31: Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$. Let $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$, x varies from 0 to ∞ and y also varies from 0 to ∞. Hence, the region of integration is the area in the first quadrant of the (x, y) plane.

Solution: Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

To cover the region, draw a radius vector OP as shown in the Figure. By turning this radius vector from x-axis to y-axis, the region of integration can be covered.

NOTES



Hence, r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \qquad I = \int_0^{\pi/2}\int_0^{\infty} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)} r\, dr\, d\theta$$

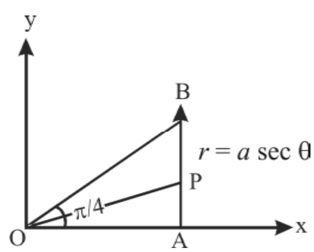
$$= \int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\infty} e^{-r^2} r\, dr = \int_0^{\pi/2} d\theta \int_0^{\infty} d\left(\frac{e^{-r^2}}{-2}\right)$$

$$= [\theta]_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \frac{\pi}{2}\left[0 - \left(\frac{-1}{2}\right)\right] = \frac{\pi}{4}$$

Example 4.32: Evaluate $\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ by changing to polar coordinates.

Solution: Let $I = \int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2}$



The region of integration is the triangle OAB (refer Figure).

This region is covered by turning the radius vector OP from OA to OB.

At O, $r = 0$, and at P, $r = a \sec\theta$

Since P lies on the line $x = a$, i.e., $r\cos\theta = a$, θ varies from 0 to $\frac{\pi}{4}$.

$$I = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r\cos\theta\, r\, dr\, d\theta}{r^2}$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \cos\theta\, dr\, d\theta = \int_0^{\pi/4} \cos\theta\, [r]_0^{a\sec\theta} d\theta$$

$$= a\int_0^{\pi/4} d\theta = a[\theta]_0^{\pi/4} = \frac{a\pi}{4}$$

NOTES

4.9.3 Evaluation of Area Using Double Integrals

Double integrals are evaluated as repeated integrals.

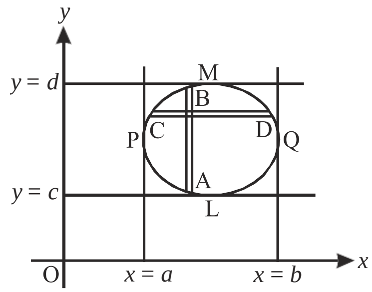


Fig. 4.5 *Closed Curve C*

Let L and M be the points on C having minimum and maximum ordinates (say c, d) and let P and Q be the points on C having the minimum and maximum abscissae (say a, b) as shown in Figure 4.5.

Let $x = \phi_1 (y)$ and $x = \phi_2 (y)$ be the equations of curves LPM and LQM (portions of C). Let $y = g_1 (x)$ and $y = g_2(x)$ be the equations of curves PLQ and PMQ (again portions of C).

It can be shown that, if $f(x, y)$ is a continuous function of x and y in the region R, then the value of the double integral is,

$$\iint_R f(x, y)\, dA \quad or \quad \iint_R f(x, y)\, dx\, dy \qquad ...(4.1)$$

This is equal to the value of the repeated integral.

$$\int^{b} dx \int_{g_1}^{g_2(x)} f(x, y)dy \qquad ...(4.2)$$

And also the value of the repeated integral,

$$\int_c^d dy \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dx \qquad ...(4.3)$$

These results enable us to evaluate double integrals as repeated integrals. For this reason, i.e., equality of Equations (4.1), (4.2) and (4.3) can be interpreted as double integrals; in fact, they are also referred to as double integrals. Note that Equation (4.2) and (4.3) have the same value under the conditions on $f(x, y)$ specified above (continuity of $f(x, y)$ in the region R).

Notes:

1. Referring to the Figure 4.5 $\left[\int_{g_1(x)}^{g_2(x)} f(x, y)dy\right]dx$ can be interpreted as the limit L, of the sum $\sum f(x_r, y_r)\Delta A_r$, obtained by subdividing the strip AB (of width dx) into sub regions ΔA_r and the integral $\int_a^b dx\int_{g_1(x)}^{g_2(x)} f(x, y)dy$ can be interpreted as the limit of sum of limits like L, obtained by considering all the strips parallel to AB and covering the region R.

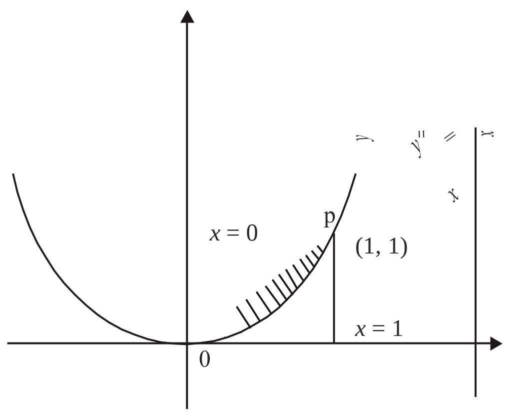
   Similarly, $\int_c^d dy\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dx$ can also be interpreted, the strips considered being parallel to CD (i.e., x-axis).

2. These interpretations are helpful in determining the limits when a double integral over an area is to be written as an equivalent repeated integral, and also in finding out the region of evaluation of a double integral given in the form of a repeated integral.
3. When the region of integration R is the rectangle bounded by $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$, the double integral $\iint_R f(x, y)dA$ is evaluated as the repeated integral

$$\int_{x_1}^{x_2} dx \int_{y_1}^{y_2} f(x, y)dy \text{ or as } \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(x, y)dx \text{ (repeated integrals with constant limits).}$$

Example 4.33: Evaluate $\int_0^1 \int_{x^2}^{x} \frac{x dy\, dx}{x^2 + y^2}$ and indicate the region of integration.

Solution: Let I denote the given integral. Then, $I = \int_0^1 dx \int_{x^2}^{x} \frac{x\, dy}{x^2 + y^2}$ written as repeated integral showing the order of integration – first with respect to y followed with respect to x). This shows that limits for integration with respect to y are determined by the curves $y = x^2$, a parabola and $y = x$, a straight line respectively. The subsequent integration is with respect to x between $x = 0$ and $x = 1$. Thus the region of integration is the area of the (x, y) plane included between the parabola $y = x^2$, the straight line $y = x$ and the ordinates $x = 0$ and $x = 1$.



Now, $$I = \int_0^1 x \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left[\tan^{-1}(1) - \tan^{-1} x \right] dx = \int_0^1 \left[\frac{\pi}{4} - \tan^{-1} x \right] dx$$

$$= \left[\frac{\pi x}{4} - \left(x \tan^{-1} x - \int x \frac{1}{1+x^2} dx \right) \right]_0^1 \text{ (Integrating by parts)}$$

$$= \left[\frac{\pi x}{4} - x \tan^{-1} x + \frac{1}{2} \log(1 + x^2) \right]_0^1 = \frac{\pi}{4} - \frac{\pi}{4} + \frac{1}{2} \log 2 = \log \sqrt{2}$$

The region of integration is indicated in the Figure by shading.

NOTES

Area of a Region of Double Integration

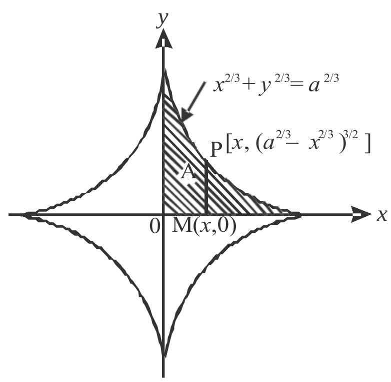
The integral $\iint_R dx\,dy$ gives the area of the region R. This is evident from the fact that $\iint_R dx\,dy$ or $\iint_R dA$ is the limit of the sum $\sum_1^n \Delta A_r$ as $n \to \infty$ and this sum is the sum of the area into which R is subdivided.

Example 4.34: Evaluate by double integration the area enclosed by the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution: Required area = 4 × Area enclosed in the first quadrant

$$= 4\iint_A dx\,dy = 4\int_0^a dx \int_0^{(a^{2/3}-x^{2/3})^{3/2}} dy$$

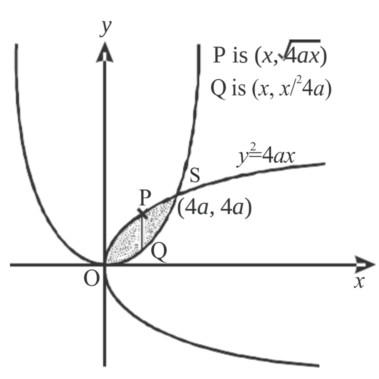
Note that the order of integration is first with respect to y and limits for y are evaluated by solving for y the boundary curves $y = 0$ and $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ in terms of x and the limits for x are the least and greatest values of x so that the ordinate at x sweeps the area in the first quadrant.



$\therefore$ Required area = $4\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx,$ which on putting $x = a\sin^3\theta$ becomes,

$$= 4\int_0^{\frac{\pi}{2}} 3a^2 \sin^2\theta\cos^4\theta\, d\theta = 12a^2 \cdot \frac{1.3.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{3}{8}\pi a^2$$

Example 4.35: By double integration, evaluate the area enclosed by the parabola $y^2 = 4ax$ and $x^2 = 4ay$



NOTES

Solution: The two parabolas are shown in the Figure.

To find the points at which the parabola intersect, we solve the equation $y^2 = 4ax$ and $x^2 = 4ay$. From the second, $y = \frac{x^2}{4a}$ which, on substitution into the first gives,

$$\frac{x^4}{16a^2} = 4ax \text{ or } x^4 - 64a^3x = 0$$

i.e., $x(x^3 - 16a^3) = 0$

i.e., $x(x-4a)(x^2 + 4ax + 16a^2) = 0$

$\therefore$ $x = 0$, $x = 4a$;

$x^2 + 4ax + 16a^2 = 0$ does not give any real value for x.

$\therefore$ Points of intersection are (0, 0), $(4a, 4a)$ (refer Figure)

Required area = $\int_A \int dx\, dy = \int_0^{4a} dx \int_{x^2/4a}^{\sqrt{4ax}} dy$

Note that the order of integration is first with respect to y followed by integration with respect to x. Limits for y are the y coordinates of Q and P (refer Figure), i.e., $\frac{x^4}{4a}$ and $\sqrt{4ax}$. Limits for x are the minimum and maximum values of x so that the strip PQ sweeps the area A. These values are 0 and $4a$.

$\therefore$ Required area $= \int_0^{4a} [y]_{x^2/4a}^{\sqrt{4ax}} dx$

$$= \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx = \left[\frac{4}{3}\sqrt{a}x^{3/2} - \frac{x^3}{12a} \right]_0^{4a}$$

$$= \left[\frac{4}{3}\sqrt{a}.4a.\sqrt{4a} - \frac{64a^3}{12a} \right] = \frac{16}{3}a^2$$

4.9.4 Evaluation of Triple Integrals

The concepts and notation explained above can be extended to integrals of functions of three variables.

Thus, $\int_a^b \int_c^d \int_e^f g(x, y, z) dxdydx$ denotes the result of integrating $g(x, y, z)$ with respect to x (treating y and z as parameters) from e to f, integrating the result with respect to y (treating z as a parameter) between c and d and integrating that result to z between a and b. Note that $dx\ dy\ dz$ by its left to right order indicates the order of integration. The corresponding limits are taken in the reverse order, i.e., e, f for x; c, d for y and a, b for z.

Example 4.36: Evaluate $\int_0^4 \int_1^2 \int_0^3 x^2 yz \, dz \, dx \, dy$.

NOTES

Solution: Let $I = \int_0^4 \int_1^2 \int_0^3 x^2 yz \; dz \; dx \; dy$

$$= \int_0^4 dy \int_1^2 dx \int_0^3 x^2 yz \, dz = \int_0^4 dy \int_1^2 \left(\frac{x^2 y z^2}{2} \right)_0^3 dx$$

$$= \int_0^4 dy \int_1^2 \frac{9}{2} x^2 y \, dx = \int_0^4 \left(\frac{3}{2} x^3 y \right)_1^2 dy$$

$$= \int_0^4 \frac{21}{2} y \, dy = \frac{21}{2} \left(\frac{y^2}{2} \right)_0^4 dy = 84$$

Note: In the above problem, the integrated x^2yz is the product of the three functions x^2, y and z each being a function of only one variable. It can be verified that I is the product of the three integrals $\int_1^2 x^2 dx$, $\int_0^4 y \, dy$ and $\int_0^3 z \, dx$.

In general, if $f(x, y, z) = f_1(x).f_2(y).\ f_3(z)$, i.e., if $f(x, y, z)$ can be expressed as the product of three functions, each involving only one variable and if a, b, c, d, l, m are constants, then,

$$\int_l^m \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz = \int_l^m \int_c^d \int_a^b f_1(x) f_2(y) f_3(z) \, dx \, dy \, dz$$

$$= \left(\int_l^m f_3(z) \, dz \right) \left(\int_c^d f_2(y) \, dy \right) \left(\int_a^b f_1(x) \, dx \right)$$

4.9.5 Evaluation of Volume Using Triple Integrals

Triple integrals are defined in a manner similar to that of double integrals. Let $f(x, y, z)$ be a function of three real variables x, y, and z continuous over the region of space, say V, enclosed by a closed surface S. Let V be divided into subregions ΔV_i and let (x_i, y_i, z_i) be any point in ΔV_i. Let L denote the sum,

$$\sum_i f\left(x_i, y_i, z_i\right) \Delta V_i$$

Under the condition on $f(x, y, z)$ stated above, the limits of the sum L can be shown to tend to a limit as the number of subregions tends to infinity in such a way that each subregion shrinks to a point and that the limit is independent of the mode of subdivision of V and the choice of points (x_i, y_i, z_i) in V_i if $f(x, y, z)$ is continuous in the region V. This limit is called the triple integral of $f(x, y, z)$ over the region V. It is denoted as,

$$\iiint_V f(x, y, z) dV \quad \text{or} \quad \iiint_V f(x, y, z) dx \, dy \, dz$$

This triple integral can be evaluated as a repeated integral.

$$\int_{z_1}^{z_2} dz \int_{f_1(z)}^{f_2(z)} dy \int_{\phi_1(y,z)}^{\phi_2(y,z)} f(x, y, z) dx$$

or

$$\int_{x_1}^{x_2} dz \int_{h_1(x)}^{h_2(x)} dy \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z) dz$$

All these are similar as in the case of double integrals.

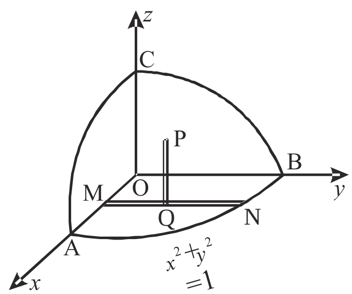
The limits of the integration can be fixed by knowing the region of integration V and its boundary S, again in a manner similar to that of double integrals.

NOTES

Finally, we note that the triple integral $\iiint_V dV$ or $\iiint_V dx\,dy\,dz$ gives the volume of the region V (enclosed by S).

Example 4.37: Evaluate $\iiint_V xyz\,dx\,dy\,dz$ taken over the positive octant of the sphere $x^2 + y^2 + z^2 = 1$.

Solution: From any point P on the surface drop PQ perpendicular to (x, y) plane. This perpendicular is moved from M, a point on the x-axis to N, a point on the curve of intersection of the sphere and (x, y) plane. This process generates a plane. This plane is moved from O to A to cover the whole region enclosed by the sphere, in the positive octant.



At P, $z = \sqrt{1 - x^2 - y^2}$ and at Q, $z = 0$

At M, $y = 0$ and at N, $y = \sqrt{1 - x^2}$

At O, $x = 0$ and at A, $x = 1$

Hence, $\iiint xyz\,dx\,dy\,dz = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

The order of integration adopted and the corresponding method of fixing the *limits* are to be noted carefully.

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\frac{xyz^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{2} xy(1 - x^2 - y^2)\,dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^2 y - xy^3)\,dy\,dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{xy^2}{2} - \frac{x^3 y^2}{2} - \frac{xy^4}{4} \right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{x(1-x^2) - x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4} \right] dx = \frac{1}{48}$$

4.10 FOURIER SERIES

NOTES

Drichlet's Conditions

Let the function $f(x)$ be defined in the interval $(c, c+2l)$. This function can be expanded as an infinite trigonometric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right),$$

if the following conditions called Drichlet's conditions are satisfied.

1. $f(x)$ is single valued, periodic with period $2l$ and finite in $(c, c+2l)$.
2. $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuities in $(c, c+2l)$.
3. $f(x)$ can have finite number of maxima and minima in the given range.

Following are the values of certain definite integrals which you require in deriving Fourier series.

If m and n ar positve integers or zeros,

1. $$\int_c^{c+2\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0 \qquad \text{...(4.4)}$$

2. $$\int_c^{c+2\pi} \sin nx\,dx = \left[\frac{-\cos nx}{n}\right]_c^{c+2\pi}$$

$$= -\frac{1}{n}\left[\cos(2n\pi + nc) - \cos nc\right]$$

$$= -\frac{1}{n}\left[\cos nc - \cos nc\right] = 0 \qquad \text{...(4.5)}$$

3. $$\int_c^{c+2\pi} \cos mx \cos nx\,dx = \frac{1}{2}\left[\int_c^{c+2\pi} (\cos(m+n)x + \cos(m-n)x)dx\right] = 0 \text{ by (4.4)} \qquad \text{...(4.6)}$$

4. $$\int_c^{c+2\pi} \sin mx \cos nx\,dx = \frac{1}{2}\left[\int_c^{c+2\pi} (\sin(m+n)x + \sin(m-n)x)dx\right] = 0 \text{ by (4.5)} \qquad \text{...(4.7)}$$

5. $$\int_c^{c+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\left[\int_c^{c+2\pi} (\cos(m-n)x - \cos(m+n)x)dx\right] = 0 \text{ by (4.4)} \qquad \text{...(4.8)}$$

Results (4.6), (4.7) and (4.8) are for $m \neq n$.

6. If $m = n$ in result (4.6) and $n \neq 0$, then you have

$$\int_c^{c+2\pi} \cos^2 nx\,dx = \frac{1}{2}\int_c^{c+2\pi} (1 + \cos 2nx)dx = \pi \qquad \text{...(4.9)}$$

7. If $m = n$ in result (4.7) and $n \neq 0$, then you have

$$\int_c^{c+2\pi} \sin^2 nx\,dx = \frac{1}{2}\int_c^{c+2\pi} (1 - \cos 2nx)dx = \pi \qquad \text{...(4.10)}$$

Change of Interval

You have seen the Fourier expansion of $f(x)$ defined in the interval $(c, c + 2\pi)$. In practice, we often require to find a Fourier series expansion of a function $f(x)$ which is not of length 2π but some other interval, say $2l$.

NOTES

You know that Fourier series expansion of $f(x)$ in $(c, c + 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx dx$ for $n = 0, 1, 2, 3,...$ and

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx\, dx \text{ for } n = 1, 2, 3,...$$

To expand $f(x)$ as Fourier series in the interval $(-l, l)$ let us define a new variable $y = \frac{\pi x}{l} \therefore x = \frac{ly}{\pi}$ and limits,

when $x = -l$ $y = -\pi$

and $x = l$ $y = \pi$

Hence, the function $f\left(\frac{yl}{\pi}\right)$ is defined in the interval $(-\pi, \pi)$ and corresponding Fourier series is,

$$f\left(\frac{yl}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny) \quad ...(4.11)$$

where $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{yl}{\pi}\right)\cos ny\, dy$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{yl}{\pi}\right)\sin ny\, dy$$

To find the Fourier series of $f(x)$ in $(-l, l)$ you revert to the variable x

i.e., $y = \frac{\pi x}{l} \therefore x = \frac{yl}{\pi}$ and $dy = \frac{\pi}{l} dx$

Then Equation (4.11) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$$

with

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\, dx \text{ for } n = 0, 1, 2,..... \text{and}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\, dx$$

If $f(x)$ is an even function $b_n = 0$

$$\therefore \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x$$

$$\text{where} \qquad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi}{l}x\,dx \text{ for } n = 0, 1, 2, \ldots.$$

If $f(x)$ is an odd function $a_n = 0$

$$\therefore \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

$$\text{where} \qquad b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi}{l}x\,dx \text{ for } n = 1, 2, 3, \ldots.$$

Example 4.38: Find the Fourier series of period $2l$ for the function $f(x) = x(2l - x)$ in $(0, 2l)$

Deduce the sum of $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \ldots.$

Solution:

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \qquad \ldots(1)$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{\sin\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right) - (2l - 2x)\left(\frac{-\cos\frac{n\pi}{l}x}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi}{l}x}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\frac{l^2}{n^2\pi^2}\left[2(l - x)\cos\frac{n\pi}{l}x\right]_0^{2l}$$

$$= \frac{2l}{n^2\pi^2}\left[-2l\cos 2n\pi - 2l\right] = -\frac{4l^2}{n^2\pi^2}$$

a_0 cannot be deduced from a_n. So

$$a_0 = \frac{1}{l}\int_0^{2l} x(2l - x)dx = \frac{1}{l}\left[\frac{2l\,x^2}{2} - \frac{x^3}{3}\right]_0^{2l} = \frac{4}{3}l^2$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{1}{l}\int_0^{2l} x(2l - x)\sin\frac{n\pi}{l}x\,dx$$

NOTES

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{-\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}\right) - (2l-2x)\left(\frac{-\sin\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi}{l}x}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^{2l}$$

$$= 0 - 2\left(\frac{l^3}{n^3\pi^3}\right)(1-1) = 0$$

Using the values of a_0, a_n, b_n in Equation (1), we get

$$f(x) = \frac{2}{3}l^2 - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{l}x \qquad \text{....(2)}$$

To deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ......\infty$ put $x = l$ in Equation (2)

$$l(2l-l) = \frac{2}{3}l^2 - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi$$

$$l^2 - \frac{2}{3}l^2 = \frac{-4l^2}{\pi^2}\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + ...\infty\right]$$

$$\frac{l^2}{3}\times\left(\frac{-\pi^2}{4l^2}\right) = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + ...\infty$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + ...\infty$$

Example 4.39: Find the Fourier series expansion of $f(x) = |x|, -1 < x < 1$

Solution:

Here, $\quad 2l = 2 \Rightarrow l = 1$

Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}x + b_n\sin\frac{n\pi}{l}x\right)$

$f(x) = |x|$ is an even function in the interval (–1, 1)

So $b_n = 0$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{l}x \qquad \text{...(1)}$$

$a_0 = \frac{1}{l}\int_{-1}^{1} f(x)dx = 2\int_0^1 f(x)\,dx$, since $f(x)$ is an even function and $l = 1$

$$= 2\int_0^1 x\,dx = 1$$

$$a_n = \frac{1}{l}\int_{-1}^{1} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$= 2\int_{-1}^{1} f(x)\cos n\pi x\,dx \quad \text{(Since } f(x) \text{ is an even function and } l = 1)$$

NOTES

$$= 2\int_{0}^{1} x\cos n\pi x\,dx$$

$$= 2\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - \left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{2}{n^2\pi^2}\left[\cos n\pi x\right]_0^1 = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

= 0 when n is even.

$= \dfrac{-4}{n^2\pi^2}$ when n is odd.

Substituting a_0, a_n in Equation (1)

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2\pi^2}\right)\cos n\pi x$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right]$$

Example 4.40: Expand $f(x)$ in Fourier series in the interval (–2, 2) when

$f(x) = 0 \qquad -2 < x < 0$

$f(x) = 1 \qquad 0 < x < 2$

Solution:

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right) \qquad \ldots(1)$$

Here, $2l = 4 \therefore l = 2$

where

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\int_{0}^{2}\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left[\frac{\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{1}{2}\times\frac{2}{n\pi}\left[\sin\frac{n\pi}{2}x\right]_0^2 = \frac{1}{n\pi}\left[\sin n\pi - \sin 0\right]$$

Though $a_n = 0$, a_0 may exist.

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0\,dx + \frac{1}{2}\int_{0}^{2} 1\,dx = 0 + \frac{1}{2}[x]_0^2 = 1$$

NOTES

$$b_n = \frac{1}{2}\int_{-2}^{0} f(x)\sin\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} \sin\frac{n\pi}{2}x\,dx$$

$$= \frac{1}{2}\left[\frac{-\cos\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]_0^2 = \frac{-1}{n\pi}\left[\cos\frac{n\pi}{2}x\right]_0^2$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = 0 \text{ if } n \text{ is even}$$

$$= \frac{2}{n\pi} \text{ if } n \text{ is odd}$$

$$\therefore \quad f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5,...}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}x$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + ...\right]$$

4.11 APPLICATIONS OF INTEGRATION IN ECONOMICS

This section will discuss the applications of integration in economics.

4.11.1 Marginal Revenue and Marginal Cost

In this section, we take up various examples to illustrate how integration proves helpful in different problems relating to Commerce and Economics.

Example 4.41: Suppose the marginal cost of a product is given by $25 + 30x - 9x^2$ and fixed cost is known to be 55. Find the total cost and average cost functions.

Solution. We know that

$$MC = \frac{d}{dx}(TC).$$

Thus, $TC = \int MC\,dx + k$

$\Rightarrow \quad TC = \int (25 + 30x - 9x^2)\,dx + k$

$\Rightarrow \quad TC = 25x + 15x^2 - 3x^3 + k.$

Since, fixed cost is 55 and $TC = FC$ when $x = 0$,

(total cost is the fixed cost or initial cost when number of units produced is zero),

we find that $55 = k$.

Thus, $TC = 25x + 15x^2 - 3x^3 + 55$

AC by definition is $\frac{TC}{x} = 25 + 15x - 3x^2 + \frac{55}{x}$.

Example 4.42: If the marginal revenue is given by $15 - 2x - x^2$, find the total revenue and demand function. Find also the maximum revenue.

Solution. We know that,

NOTES

$$MR = \frac{d}{dx}(TR)$$

$$\Rightarrow \quad TR = \int MR\, dx + k$$

Thus, $TR = \int (15 - 2x - x^2)\, dx + k = 15x - x^2 - \frac{x^3}{3} + k$.

At $x = 0$, $TR = 0$, and thus $k = 0$.

Hence, $TR = 15x - x^2 - \frac{x^3}{3}$.

If p is the demand function, then

$TR = px$ (definition)

$$\Rightarrow \quad p = \frac{TR}{x} = 15 - x - \frac{x^2}{3}.$$

Again, for maximum revenue

$$\frac{d}{dx}(TR) = 0 \quad \Rightarrow \quad 15 - 2x - x^2 = 0$$

$$\Rightarrow \quad x = -5, 3.$$

Since, $x = -5$ is not possible, we take $x = 3$.

Now, $\frac{d^2(TR)}{dx^2} = -2 - 2x$

$$\therefore \quad \left(\frac{d^2}{dx^2}(TR)\right)_{x=3} = -2 - 6 = -8 < 0$$

$\Rightarrow$ there is a max. at $x = 3$.

i.e., revenue is max. when $x = 3$.

Also then, maximum revenue $= 15 \times 3 - 9 - \frac{27}{3} = 27$.

Example 4.43: ABC Co. Ltd. has approximated the marginal revenue function for one of its products by $MR = 20x - 2x^2$. The marginal cost function is approximated by $MC = 81 - 16x + x^2$.

Determine the profit maximizing output and the total profit at the optimal output.

Solution. Profit π is maximum if $MR = MC$

i.e., if $20x - 2x^2 = 81 - 16x + x^2$

or, $3x^2 - 36x + 81 = 0$

or, $x^2 - 12x + 27 = 0$

or, $(x - 3)(x - 9) = 0$

$x = 3, 9.$

NOTES

For max. profit, we should have, $\dfrac{d^2\pi}{dx^2} < 0$

i.e., $\dfrac{d^2R}{dx^2} < \dfrac{d^2C}{dx^2}$

$\Rightarrow$ $\dfrac{d}{dx}(MR) < \dfrac{d}{dx}(MC)$

i.e., $20 - 4x < -16 + 2x$

or, $6x - 36 > 0$

or, $x > 6.$

Thus, we take $x = 9$ (out of the two values of x) for maximum profit.

Now, profit $\pi = R - C$

So, at $x = 9$, profit $= \int_0^9 \dfrac{d}{dx}(R - C)dx = \int_0^9 \left(\dfrac{dR}{dx} - \dfrac{dc}{dx}\right)dx$

$$= \int_0^9 (MR - MC)dx$$

$$= \int_0^9 (20x - 2x^2 - 81 + 16x - x^2)dx$$

$$= \int_0^9 (-3x^2 + 36x + 81)dx$$

$$= \left[-x^3 + 18x^2 - 81x\right]_0^9$$

$$= -729 + 1458 - 729 = 0.$$

Thus, profit maximizing value is 9 and the total profit is zero.

Note: We have used the definite integral idea above. We could also proceed as:

$$\pi = TR - TC = \int MR - \int MC$$

$$= \int (20x - 2x^2)dx - \int (81 - 16x + x^2)dx$$

$$= \int (-3x^2 + 36x + 81)dx$$

$\Rightarrow$ Profit $= -x^3 + 18x^2 - 81x$

and thus profit at the optimal output 9, is

$-9^3 + 18.9^2 - 18.9 = -729 + 1458 - 729 = 0.$

Example 4.44: The marginal cost function of manufacturing x pairs of shoes is $6 + 10x - 6x^2$. The total cost of producing a pair of shoes is Rs 12. Find the total average cost function.

Solution. We have,

$$MC = 6 + 10x - 6x^2$$

$\Rightarrow$ $TC = \int (6 + 10x - 6x^2)dx + k$

$$= 6x + 5x^2 - 2x^3 + k.$$

NOTES

For one pair of shoes $TC = 12$

i.e., for $x = 1,\ TC = 12$

Thus, $12 = 6 + 5 - 2 + k \Rightarrow k = 3.$

Hence, $TC = 6x + 5x^2 - 2x^3 + 3$

Again $AC = \frac{TC}{x} = 6 + 5x - 2x^2 + \frac{3}{x}$

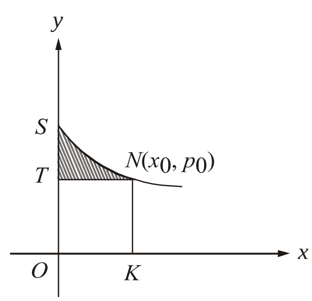
which gives the average cost function.

4.11.2 Consumer and Producer Surplus

Suppose, p is the price that a consumer is willing to pay for a quantity x of a certain commodity, then p and x are related to each other through the demand function and we express this by saying that $p = f(x)$. The graph of this is generally sloping downwards as demand decreases when price is increased (with increase in price, the consumer is inclined to buy less).

Again, suppose now that p is the price that a producer wishes to charge for selling a quantity x of a particular commodity. Then, p and x are related to each other through what is called the *supply curve* $p = g(x)$. This is generally sloping upwards as when the price increases, the producer is inclined to supply more.

If the two curves (supply and demand) intersect, we say *economic equilibrium* is attained. The point of intersection is then called the *equilibrium point*. It is, of course, not essential that the two curves intersect (*i.e.*, economic equilibrium is achieved).



If the point of intersection N has coordinates (x_0, p_0) then p_0 (the market price) is the price which both the consumer and the producer are ready to pay and accept respectively for the quantity x_0 of the commodity. The total revenue in that case is $p_0 \times x_0$.

Sometimes, it happens that a consumer is ready to pay, say, Rs 50 for a certain commodity but gets it for, say, Rs 40 in the market and thus earns (saves) Rs 10. This gain to the consumer is termed as the *consumer surplus*. It is shown by the shaded portion in Figure 4 and is given by the formula

$$CS = \int_0^{x_0} f(x)dx - (p_0 \times x_0)$$

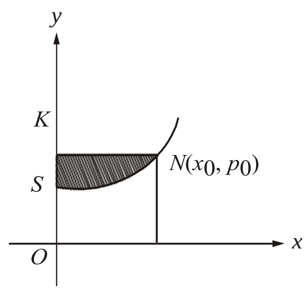
where, of course, we know the integral $\int_0^{x_0} f(x)dx$ represents the area enclosed by the curve $p = f(x)$, the x-axis and the ordinate NK $(x = x_0)$ *i.e*, the area *STOKNS* in the figure. This is, in fact, total revenue that would have been generated because of the willingness of some consumers to pay more.

Again $p_0 \times x_0$ is the area of the rectangle *TOKN* and represents the actual revenue achieved. The difference is thus the surplus.

Similarly, sometimes there are producers who are willing to charge less than the market price (to increase sales) which the consumer actually pays. The gain of this to the producer is called Producer Surplus (*PS*). It is shown by the shaded portion in the Figure 5 and is given by the formula.

$$PS = (p_0 \times x_0) - \int_0^{x_0} g(x)dx$$

where, $p = g(x)$ is the supply curve.



It is the difference of the total revenue actually achieved and the revenue that would have been generated by the willingness of some producers to charge less.

Example 4.45: Given the demand function $p = 45 - \frac{x}{2}$ find consumer surplus when $p_0 = 32.5, x_0 = 25.$

Solution. We have,

$$CS = \int_0^{25}\left(45 - \frac{x}{2}\right)dx - (32.5) \times 25$$

$$= \left\{45x - \frac{x^2}{4}\right\}_0^{25} - 812.5$$

$$= 45 \times 25 - \frac{25 \times 25}{4} - 812.5$$

$$= 156.25$$

Example 4.46: Given the demand function $p_d = 4 - x^2$ and the supply function $p_s = x + 2$. Find CS and PS (assuming pure competition).

Solution. For market equilibrium, $p_s = p_d$

$$\Rightarrow \qquad x + 2 = 4 - x^2$$

$$\Rightarrow \qquad x = -2, 1$$

Since, –ve value of x is not possible, we have, $x = 1$.

Since, for $x = 1$, $p = 3$, we have, $x_0 = 1$, $p_0 = 1$.

Hence,

$$CS = \int_0^1 (4 - x^2)dx - 3 = \left\{4x - \frac{x^3}{3}\right\}_0^1 - 3 = \frac{2}{3}.$$

Also

$$PS = 3 - \int_0^1 (x+2)dx = 3 - \left\{ \frac{x^2}{2} + 2x \right\}_0^1 = \frac{1}{2}.$$

NOTES

which give the required values.

Example 4.47: Under a monopoly, the quantity sold and market price are determine by the demand function. If the demand function for a profit maximizing monopolist is $p = 274 - x^2$ and $MC = 4 + 3x$, find CS.

Solution. We are given that $p = 274 - x^2$.

$$\Rightarrow \quad TR = p \times x = 274x - x^3$$

$$\Rightarrow \quad MR = \frac{d}{dx}(TR) = 274 - 3x^2$$

Now, the monopolist maximizes profit at

$$MR = MC.$$

i.e., $274 - 3x^2 = 4 + 3x$

$$\Rightarrow \quad 3(x^2 + x - 90) = 0$$

$$\Rightarrow \quad x = 9, -10$$

Since, $x = -10$ is not possible, we have,

$$x_0 = 9, \text{ also then } p_0 = 193.$$

Hence,

$$CS = \int_0^9 (274 - x^2)dx - 193 \times 9 = 486.$$

Example 4.48: Find the consumer surplus at equilibrium price, if the demand function is $D = \frac{25}{4} - \frac{p}{8}$ and supply function is $p = 5 + D$.

Solution. We have, $p = 5 + D$

$$= 5 + \left(\frac{25}{4} - \frac{p}{}\right)$$

$$\Rightarrow \quad p = 10$$

and thus $D = 5$.

So the equilibrium price $p = 10$, and $D = 5$.

Hence, $CS = \int_0^5 (50 - 8D)dD - 10 \times 5, \quad \left[D = \frac{25}{4} - \frac{p}{8} \Rightarrow p = 50 - 8D \right]$

$$= \left\{ 50D - \frac{8D^2}{2} \right\}_0^5 - 50$$

$$= 250 - 150 - 50 = 100$$

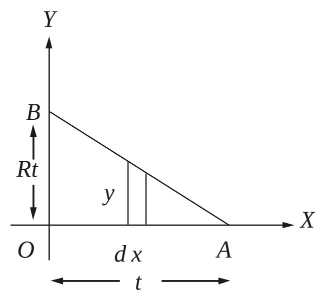
which is the required value.

4.11.3 Economic Lot Size Formula

NOTES

In an earlier chapter we discussed inventory control problems assuming that amount of inventory remains same throughout the production run. But in actual practice, since goods are being sold all through, the amount of inventory goes on decreasing and so the cost of keeping it also decreases, We discuss now this type of situation.

Suppose, a contractor has an order of supplying goods at a uniform rate R per unit of time. His one production run takes t units of time where t is supposed to be fixed for each production. We assume that production time is negligible and so there is no delay in fulfilling the demand as long as a new run is started whenever inventory is zero. The zero inventory, in fact, is a signal for the *start* of next production run. The cost of holding inventory is proportional to the amount of inventory and the time for which it is kept. Suppose time is measured along x-axis and inventory along y-axis.



In the beginning, the inventory is Rt and at the end of a production run it is zero. Let B be the point $(0, Rt)$ and A be $(t, 0)$.

Suppose at any instant x, inventory is y.

We can safely assume that for small change in time, say, dx, it remains same.

The cost of holding y units of inventory for dx units of time will be equal to $c_1\, y\, dx$, where, c_1 is the fixed cost of holding one unit of inventory for one unit of time.

The cost of holding inventory throughout a production

$$= \int_0^t c_1 y\, dx = c_1 \int_0^t y dx \quad (c_1 \text{ being fixed is constant})$$

Equation of line AB is $\quad \frac{x}{t} + \frac{y}{Rt} = 1$

or, $\quad y = Rt - xR$

Thus, cost of holding inventory Rt

$$= c_1 \int_0^t (Rt - xR)dx$$

$$= c_1 \left[Rtx - \frac{x^2}{2} R \right]_0^t$$

$$= \frac{1}{2} c_1 Rt^2 .$$

Suppose, now that c_2 is the cost of step-up per production run, then total cost

$$c = \frac{1}{2} c_1 Rt^2 + c_2$$

Hence, average cost $AC = \frac{1}{2} c_1 Rt + \frac{c_2}{t}$.

For AC to be max. or min.

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$$\frac{dA}{dt} = 0.$$

or, $$\frac{1}{2} c_1 R - \frac{c_2}{t_2} = 0$$

or, $$t = \sqrt{\frac{2c_2}{c_1 R}}$$

Since, for this value of t, $\frac{d^2 A}{dt^2} = \frac{2c_2}{t^3} > 0$.

The value gives a min.

Hence, $t = \sqrt{\frac{2c_2}{c_1 R}}$ gives minimum cost.

The quantity produced q, in one production run is Rt.

Hence, $q = Rt$

$$\Rightarrow \quad q = \sqrt{\frac{2c_2}{c_1} R}$$

for minimum cost.

This quantity produced, *i.e.*, $\sqrt{\frac{2c_2}{c_1} R}$ is called optimum run size and the equation.

$$q = \sqrt{\frac{2c_2}{c_1} R}$$

is called *Economic Lot Size Formula.*

The minimal average cost

$$= \frac{1}{2} c_1 R \sqrt{\frac{2c_2}{c_1 R}} + c_2 \sqrt{\frac{c_1 R}{2c_2}} = \sqrt{2c_1 c_2 R}$$

Check Your Progress

11. Write the sign conventions followed in finding the area under the curve.
12. How can you find the area of a region of double integration?
13. Define integrals of functions of three variables.
14. What is the Fourier series in the expansion of $f(x)$ in $(c, c + 2\pi)$?

4.12 SUMMARY

- Integration is the reverse process of differentiation. Differentiation and integration cancel each other.
- Integral of sum/difference of two functions is equal to the sum/difference of integral of the two functions.
- In the method of substitution, we express the given integral in terms of another integral in which the independent variable x is changed to another variable t through some suitable relation $x = \phi(t)$.
- If $f(x)$ is a function such that $\int f(x)\, d(x) = g(x)$ then the definite integral $\int_a^b f(x)dx$ is defined by $\int_a^b f(x)dx = \{g(x)\}_a^b = g(b) - g(a)$ where, a and b are

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two real numbers, and are called respectively, the lower and the upper limits of the integral.

- Partial fractions are used to find the integrals of rational functions while substitution is used to find the integrals of irrational functions.
- If the integrand consists of even powers of x only, then the substitution $x^2 = t$ is helpful while resolving into partial fractions.
- The area under the curve $y = f(x)$ between the limits a, b can be written as a definite integral, $\int_a^b f(x)dx = F(b) - F(a)$.
- If the limit of the sum $S = \sum_{r=1}^{n} f(x_r, y_r)\Delta A_r$ exists, as $n \to \infty$ and as each sub region $\Delta A_r \to 0$, and the limit is independent of the manner in which the region R is subdivided and the points (x_r, y_r) chosen in the region A_r, then that limit is called the double integral of $f(x, y)$ over the region R.
- The evaluation of certain double integrals becomes easier by effecting a change in the variables. Double integrals are evaluated as repeated integrals.
- $\int_a^b \int_c^d \int_e^f g(x, y, z)\, dxdydz$ denotes the result of integrating $g(x, y, z)$ with respect to x (treating y and z as parameters) from e to f, integrating the result with respect to y (treating z as a parameter) between c and d and integrating that result to z between a and b.
- If the function $f(x)$ is defined in the interval $(c, c + 2l)$ then this function can be expanded as an infinite trigonometric series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{l}x + b_n \sin\frac{n\pi}{l}x\right)$ if the Drichlet's conditions are satisfied.
- Drichlet's conditions are- $f(x)$ is single valued, periodic with period $2l$ and finite in $(c, c + 2l)$; $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuities in $(c, c + 2l)$; $f(x)$ can have finite number of maxima and minima in the given range.

4.13 KEY TERMS

- **Integrand:** If $f(x)$ is the differential with respect to x of a function g(x) then $f(x)$ is called the integrand
- **Indefinite integral:** When we are not giving a definite value to the integral, then the integral is referred to as indefinite integral
- **Definite integral:** When we give the lower and upper limits to the integral which are both real numbers, then it is referred to as definite integral
- **Rational function:** A function of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials in x, is called a rational function

4.14 ANSWERS TO 'CHECK YOUR PROGRESS'

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1. If $\frac{d}{dx} g(x) = f(x)$

   Then, $\int f(x)\, dx = g(x) + c$

   Where c is some constant, called the constant of integration.

2. Let $\frac{d}{dx} g(x) = f(x)$

   Then, $\int f(x)\, dx = g(x)$ [By definition]

   $\Rightarrow \quad \frac{d}{dx}[f(x)\, dx] = \frac{d}{dx}[g(x)] = f(x)$

   Which proves the result.

3. Suppose $f(x)$ is a function such that

   $\int f(x)\, dx = g(x)$

   The definite integral $\int_a^b f(x)\, dx$ is defined by

   $\int_a^b f(x)\, dx = \{g(x)\}_a^b = g(b) - g(a)$

   where, a and b are two real numbers, and are called respectively, the lower and the upper limits of the integral.

4. A definite integral equals zero when the limits of integration are identical,

   $$\int_a^a f(x)\, dx = [f(x)]_a^a = f(a) - f(a) = 0$$

   The area on a single point is zero because the width dx of the rectangle, is zero.

5. The directed length of the interval of integration is given by,

   $$\int_a^b dx = b - a$$

6. If one of the limits is the variable itself, the definite integral becomes equal to the indefinite integral of the function,

   $$\int_a^x f(x)dx = f(x) - f(a) = f(x) + C$$

   Where $C = -f(a)$ is a constant.

7. Integral of the product of two functions

   = First function × Integral of the second – Integral of (Differential of first × Integral of the second function).

8. One thumb rule may be followed by remembering a keyword 'ILATE'. I-means inverse function, L-means logarethmic, A-means algebraic, T-means trigonometric and E-means exponential.

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9. A function of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials in x, is called a rational function.
10. If the integrand consists of even powers of x only, then the substitution $x^2 = t$ is helpful while resolving into partial fractions.
11. If the function $y = f(x)$ is positive in the interval (a, b) and the curve is above the x-axis then $\int_a^b f(x)dx$ is positive.

    If $y = f(x)$ is negative in the interval (a, b) and the curve is below x-axis then, $\int_a^b f(x)dx$ is negative.

    If $y = f(x)$ changes sign in the interval and the curve crosses the x-axis, the area is the algebraic sum of a positive area and a negative area.
12. The integral $\iint_R dx\,dy$ gives the area of the region R. This is evident from the fact that $\iint_R dx\,dy$ or $\iint_R dA$ is the limit of the sum $\sum_1^n \Delta A_r$ as $n \to \infty$ and this sum is the sum of the area into which R is subdivided.
13. $\int_a^b \int_c^d \int_e^f g(x, y, z)dxdydx$ denotes the result of integrating $g(x, y, z)$ with respect to x (treating y and z as parameters) from e to f, integrating the result with respect to y (treating z as a parameter) between c and d and integrating that result to z between a and b. Note that $dx\,dy\,dz$ by its left to right order indicates the order of integration. The corresponding limits are taken in the reverse order, i.e., e, f for x; c, d for y and a, b for z.
14. Fourier series expansion of $f(x)$ in $(c, c + 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx dx$ for $n = 0, 1, 2, 3,...$ and

$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\, dx$ for $n = 1, 2, 3,...$

4.15 QUESTIONS AND EXERCISES

Short-Answers Questions

1. What is the relation between integration and differentiation?
2. Define constant of integration.
3. Define definite integrals.
4. What are indefinite integrals?

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5. What is the method of substitution?
6. How is the integration of rational and irrational functions done?
7. Write some applications of integrals.
8. Define double integral.
9. How do you find area using triple integral?
10. What are Drichlet's conditions?

Long-Answers Questions

1. Integrate the following functions with respect to x:

   (i) $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$ (ii) $\frac{1}{\sqrt{x-1}-\sqrt{x+1}}$

2. Evaluate the following integrals:

   (i) $\int_0^{\pi/2} \sin x \, dx$ (ii) $\int_0^{\pi/4} \sin^2 x \, dx$

   (iii) $\int_2^3 \frac{1}{x} dx$ *(iv)* $\int_2^3 (x+1)^2 \, dx$

   (v) (v) $\int_0^1 \frac{1}{1+x^2} \, dx$ (vi) $\int_a^b x^2 \, dx$

3. Evaluate $\int \frac{1}{x^2\sqrt{1-x^2}} \, dx$

4. Evaluate using Integration (i) $\frac{1-\tan x}{1+\tan x}$ (ii) $\frac{1}{\sqrt{x+x}}$ *(iii)* $\frac{\sec x \operatorname{cosec} x}{\log \tan x}$

5. Integrate, (i) $\frac{x^2}{\sqrt{x^2-1}}$ (ii) $\frac{x^3}{\sqrt{x^8+1}}$ (iii) $\frac{1}{(\sqrt{a^2-x^2})^3}$

6. Show that $\int \frac{x^2+2x+3}{\sqrt{x^2+1}} dx = \frac{5}{2}\left[\sqrt{x^2+1} + \sin h^{-1} x\right]$

7. Prove the following:

   (i) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

   (ii) $\int_p^q f(x)dx = \int_p^q f(p+q-x)dx$

   (iii) $\int_0^a f(a+x)dx = \int_0^a f(-x)dx$

8. Find the area under the curve:

   (i) $y = x^2 + 4x + 5, -2 \le x \le 1$

   (ii) $y = \frac{x^2}{2} + 1, 0 \le x \le 4$

   (iii) $y = 9 - x^2, 1 \le x \le 3$

   (iv) $y = a^2 - x^2, 0 \le x \le 1$

9. Evaluate the following by changing to polar coordinates.

(i) $\int_0^a \int_y^a \frac{x^2 dxdy}{(x^2+y^2)^{3/2}}$ (ii) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dxdy$

(iii) $\int_0^\infty \int_0^\infty \frac{dxdy}{(x^2+y^2+a^2)^2}$ (iv) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\, dx\, dy}{x^2+y^2}$

(v)(v) $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dy\, dx$

10. Evaluate $\iint (x+y)^2\, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

11. Find, by double integration, the area between the parabola $y^2 = 4ax$ and the line $y = x$.

12. Prove that $\iint (x^2+y^2) dx\, dy$, evaluated over the region R formed by the lines $y = 0$, $x = 1$, $y = x$ is $\frac{1}{3}$.

13. Evaluate $\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$ for all positive values of x, y, z for which the integral is real.

14. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by triple integration.

15. Given is $f(x) = \frac{-\pi}{4}$ in $-\pi < x < 0$

$= \frac{\pi}{4}$ in $0 < x < \pi$ and $f(x + 2\pi) = f(x)$ for all x.

Expand in Fourier series.

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UNIT 5 LINEAR PROGRAMMING

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Structure

5.0 INTRODUCTION

In this unit, you will learn about the use of linear programming in decision-making. For a manufacturing process, a production manager has to take decisions as to what quantities and which process or processes are to be used so that the cost is minimum and profit is maximum. Currently, this method is used in solving a wide range of practical business problems. The word 'linear' means that the relationships are represented by straight lines. The word 'programming' means following a method for taking decisions systematically.

You will understand the extensive use of Linear Programming (LP) in solving resource allocation problems, production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning, etc. Linear programming has been successfully applied in agricultural and industrial applications.

You will learn a few basic terms like linearity, process and its level, criterion function, constraints, feasible solutions, optimum solution, etc. The term linearity implies straight line or proportional relationships among the relevant variables. Process means the combination of one or more inputs to produce a particular output. Criterion function is an objective function which is to be either maximized or minimized. Constraints are limitations under which one has to plan and decide. There are restrictions imposed upon

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decision variables. Feasible solutions are all those possible solutions considering given constraints. An optimum solution is considered the best among feasible solutions.

You will also learn to formulate linear programming problems and put these in a matrix form. The objective function, the set of constraints and the non-negative constraint together form a linear programming problem. In this unit, you will also learn the methods of solving a Linear Programming Problem (LPP) with two decision variables using the graphical method. All linear programming problems may not have unique solutions. You may find some linear programming problems that have an infinite number of optimal solutions, unbounded solutions or even no solution.

Finally, you will learn about the canonical or standard form of LPP. In the standard form, irrespective of the objective function, namely maximize or minimize, all the constraints are expressed as equations. Moreover, the Right Hand Side (RHS) of each constraint and all variables are non-negative. The simplex method and *M* method are the methods of solution by iterative procedure in a finite number of steps using matrix.

5.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Understand the significance of linear programming
- Know the terms associated with a linear programming problem
- Learn how to formulate a linear programming problem
- Form a matrix of a linear programming problem
- Explain the applications and limitations of linear programming problems
- Solve a linear programming problem with two variables using the graphical method
- Describe linear programming problems in canonical form
- Solve linear programming problems using the simplex method
- Solve linear programming problems using the *M* method

5.2 INTRODUCTION TO LINEAR PROGRAMMING PROBLEM

Decision-making has always been very important in the business and industrial world, particularly with regard to the problems concerning production of commodities. Which commodity/commodities to produce, in what quantities and by which process or processes, are the main questions before a production manager. English economist Alfred Marshall pointed out that the businessman always studies his production function and his input prices and substitutes one input for another till his costs become the minimum possible. All this sort of substitution, in the opinion of Marshall, is being done by businessman's trained instinct rather than with formal calculations. But now there does exist a method of formal calculations often termed as Linear Programming. This method was first formulated by a Russian mathematician L.V. Kantorovich, but it was developed later in 1947 by George B. Dantzig 'for the purpose of scheduling the complicated procurement activities of the United States Air Force'. Today, this method is being used in solving a wide range of practical business problems. The advent of electronic computers has further increased its applications to solve many other problems in industry. It is being considered as one of the most versatile management tools.

in modern times in various fields as stated above.

5.2.1 Meaning of Linear Programming

Linear Programming (LP) is a major innovation since World War II in the field of business decision-making, particularly under conditions of certainty. The word 'Linear' means that the relationships are represented by straight lines, i.e., the relationships are of the form $y = a + bx$ and the word 'Programming' means taking decisions systematically. Thus, LP is a decision-making technique under given constraints on the assumption that the relationships amongst the variables representing different phenomena happen to be linear. In fact, Dantzig originally called it 'programming of interdependent activities in a linear structure' but later shortened it to 'Linear Programming'. LP is generally used in solving maximization (sales or profit maximization) or minimization (cost minimization) problems subject to certain assumptions. Putting in a formal way, 'Linear Programming is the maximization (or minimization) of a linear function of variables subject to a constraint of linear inequalities.' Hence, LP is a mathematical technique designed to assist the organization in optimally allocating its available resources under conditions of certainty in problems of scheduling, product-mix, and so on.

5.2.2 Fields Where Linear Programming can be Used

The problem for which LP provides a solution may be stated to maximize or minimize for some dependent variable which is a function of several independent variables when the independent variables are subject to various restrictions. The dependent variable is usually some economic objectives, such as profits, production, costs, work weeks, tonnage to be shipped, etc. More profits are generally preferred to less profits and lower costs are preferred to higher costs. Hence, it is appropriate to represent either maximization or minimization of the dependent variable as one of the firm's objective. LP is usually concerned with such objectives under given constraints with linearity assumptions. In fact, it is powerful to take in its stride a wide range of business applications. The applications of LP are numerous and are increasing every day. LP is extensively used in solving resource allocation problems. Production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning, etc., are some of its most fertile application areas. More specifically, LP has been successfully applied in the following fields:

(i) ***Agricultural Applications:*** LP can be applied in farm management problems as it relates to the allocation of resources, such as acreage, labour, water supply or working capital in such a way that is maximizes net revenue.

(ii) ***Contract Awards:*** Evaluation of tenders by recourse to LP guarantees that the awards are made in the cheapest way.

(iii) ***Industrial Applications:*** Applications of LP in business and industry are of most diverse kind. Transportation problems concerning cost minimization can be solved by this technique. The technique can also be adopted in solving the problems of production (product-mix) and inventory control.

Thus, LP is the most widely used technique of decision-making in business and industry

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5.3 COMPONENTS OF LINEAR PROGRAMMING PROBLEM

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The following are the components of linear programming problem:

5.3.1 Basic Concepts and Notations

There are certain basic concepts and notations to be first understood for easy adoption of the LP technique. A brief mention of such concepts is as follows:

(i) ***Linearity:*** The term linearity implies straight line or proportional relationships among the relevant variables. Linearity in economic theory is known as constant returns which means that if the amount of input doubles, the corresponding output and profit are also doubled. Linearity assumption, thus, implies that if two machines and two workers can produce twice as much as one machine and one worker; four machines and four workers twice as much as two machines and two workers, and so on.

(ii) ***Process and Its Level:*** Process means the combination of particular inputs to produce a particular output. In a process, factors of production are used in fixed ratios, of course, depending upon technology and as such no substitution is possible with a process. There may be many processes open to a firm for producing a commodity and one process can be substituted for another. There is, thus, no interference of one process with another when two or more processes are used simultaneously. If a product can be produced in two different ways, then there are two different processes (or activities or decision variables) for the purpose of a linear program.

(iii) ***Criterion Function:*** Criterion function is also known as objective function which states the determinants of the quantity either to be maximized or to be minimized. For example, revenue or profit is such a function when it is to be maximized or cost is such a function when the problem is to minimize it. An objective function should include all the possible activities with the revenue (profit) or cost coefficients per unit of production or acquisition. The goal may be either to maximize this function or to minimize this function. In symbolic form, let ZX denote the value of the objective function at the X level of the activities included in it. This is the total sum of individual activities produced at a specified level. The activities are denoted as $j = 1, 2, ..., n$. The revenue or cost coefficient of the jth activity is represented by C_j. Thus, $2X_1$, implies that X units of activity $j = 1$ yields a profit (or loss) of $C_1 = 2$.

(iv) ***Constraints or Inequalities:*** These are the limitations under which one has to plan and decide, i.e., restrictions imposed upon decision variables. For example, a certain machine requires one worker to be operated upon; another machine requires at least four workers (i.e., > 4); there are at most 20 machine hours (i.e., < 20) available; the weight of the product should be say 10 lbs, and so on, are all examples of constraints or why are known as inequalities. Inequalities like $X > C$ (reads X is greater than C or $X < C$ (reads X is less than C) are termed as strict inequalities. The constraints may be in form of weak inequalities like $X \leq C$ (reads X is less than or equal to C) or $X \geq C$ (reads C is greater than or equal to C). Constraints may be in the form of strict equalities like $X = C$ (reads X is equal to C).

Let b_i denote the quantity *b* of resource *i* available for use in various production processes. The coefficient attached to resource *i* is the quantity of resource *i* required for the production of one unit of product *j*.

(v) ***Feasible Solutions:*** Feasible solutions are all those possible solutions which can be worked upon under given constraints. The region comprising of all feasible solutions is referred as *Feasible Region.*

(vi) ***Optimum Solution:*** Optimum solution is the best of the feasible solutions.

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5.3.2 General Form of the Linear Programming Model

Linear Programming problem mathematically can be stated as under:

Choose the quantities,

$$X_j \geq 0 \quad (j = 1,..., n) \qquad ...(5.1)$$

This is also known as the non-negativity condition and in simple terms means that no *X* can be negative.

To maximize,

$$Z = \sum_{j=1}^{n} C_j X_j \qquad ...(5.2)$$

Subject to the constraints,

$$\sum_{j=1}^{n} a_{ij} X_j \leq b_i \qquad (i = 1,...,m) \qquad ...(5.3)$$

The above is the usual structure of a linear programming model in the simplest possible form. This model can be interpreted as a profit maximization situation where *n* production activities are pursued at level X_j which have to be decided upon, subject to a limited amount of *m* resources being available. Each unit of the *j*th activity yields a return *C* and uses an amount a_{ij} of the *i*th resource. Z denotes the optimal value of the objective function for a given system.

Assumptions or the Conditions to be Fulfilled Underlying the LP Model

LP model is based on the assumptions of proportionality, additivity, certainty, continuity and finite choices.

Proportionality is assumed in the objective function and the constraint inequalities. In economic terminology this means that there are constant returns to scale, i.e., if one unit of a product contributes ` 5 toward profit, then 2 units will contribute ` 10, 4 units ` 20, and so on.

Certainty assumption means the prior knowledge of all the coefficients in the objective function, the coefficients of the constraints and the resource values. LP model operates only under conditions of certainty.

Additivity assumption means that the total of all the activities is given by the sum total of each activity conducted separately. For example, the total profit in the objective function is equal to the sum of the profit contributed by each of the products separately.

Continuity assumption means that the decision variables are contiunous. Accordingly the combinations of output with fractional values, in case of product-mix problems, are possible and obtained frequently.

Check Your Progress

1. What is linear programming?
2. What is meant by criterion function in linear programming?
3. Mention two areas where linear programming finds application.
4. What are constraints in linear programming?
5. What is a solution in linear programming problem?
6. What is a 'basic solution' of an LPP?
7. What is basic and non-basic variables?
8. What do you understand by basic feasible solution?

Finite choices assumption implies that finite number of choices are available to a decision-maker and the decision variables do not assume negative values.

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5.4 FORMULATION OF LINEAR PROGRAMMING PROBLEM

This section will discuss the process of formulation of linear programming problem:

5.4.1 Graphic Solution

The procedure for mathematical formulation of an LPP consists of the following steps:

Step 1: The decision variables of the problem are noted.

Step 2: The objective function to be optimized (maximized or minimized) as a linear function of the decision variables is formulated.

Step 3: The other conditions of the problem, such as resource limitation, market constraints, interrelations between variables, etc., are formulated as linear inequations or equations in terms of the decision variables.

Step 4: The non-negativity constraint from the considerations is added so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative constraint together form a linear programming problem.

5.4.2 General Formulation of Linear Programming Problem

The general formulation of the LPP can be stated as follows:

In order to find the values of n decision variables $X_1, X_2, ..., X_n$ to maximize or minimize the objective function.

$$Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n \qquad \ldots (5.4)$$

$$\left.\begin{aligned} &a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \, (\leq, =, \geq) b_1 \\ &a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n \, (\leq, =, \geq) b_2 \\ &\vdots \\ &a_{i1}X_1 + a_{i2}X_2 + \cdots + a_{in}X_n \, (\leq, =, \geq) b_i \\ &\vdots \\ &a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n \, (\leq = \geq) b_m \end{aligned}\right\} \qquad \ldots (5.5)$$

Here, the constraints can be inequality $\leq$ or $\geq$ or even in the form an equation (=) and finally satisfy the non-negative restrictions:

$$X_1 \geq 0,\ X_2 \geq 0 \ldots X_n \geq 0 \qquad \ldots (5.6)$$

5.4.3 Matrix Form of Linear Programming Problem

The LPP can be expressed in the matrix form as follows:

Maximize or minimize $Z = CX \rightarrow$ Objective function

Subject to $AX\ (\leq, =, \geq)\ B \rightarrow$ Constant equation

$B > 0, X \geq 0 \rightarrow$ Non-negativity restrictions

Where, $X = (X_1, X_2, \cdots, X_n)$

$C = (C_1, C_2, \cdots, C_n)$

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_m \end{pmatrix} \quad A = \begin{pmatrix} a_{11}a_{12} \cdots a_{1n} \\ a_{21}a_{22} \cdots a_{2n} \\ \vdots \\ a_{m1}a_{m2} \cdots a_{mn} \end{pmatrix}$$

NOTES

Example 5.1: A manufacturer produces two types of models M_1 and M_2. Each model of the type M_1 requires 4 hours of grinding and 2 hours of polishing; whereas each model of the type M_2 requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. The profit on M_1 model is ` 3.00 and on model $M_2$ is ` 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?

Solution:

Decision variables: Let X_1 and X_2 be the number of units of M_1 and M_2.

Objective function: Since the profit on both the models are given, we have to maximize the profit, viz.,

Max $Z = 3X_1 + 4X_2$

Constraints: There are two constraints: one for grinding and the other for polishing.

The number of hours available on each grinder for one week is 40 hours. There are 2 grinders. Hence, the manufacturer does not have more than 2 × 40 = 80 hours for grinding. M_1 requires 4 hours of grinding and M_2 requires 2 hours of grinding.

The grinding constraint is given by,

$$4X_1 + 2X_2 \leq 80$$

Since there are 3 polishers, the available time for polishing in a week is given by 3 × 60 = 180. M_1 requires 2 hours of polishing and M_2 requires 5 hours of polishing. Hence, we have $2X_1 + 5X_2 \leq 180$

Thus, we have,

Max $Z = 3X_1 + 4X_2$

Subject to $4X_1 + 2X_2 \leq 80$

$$2X_1 + 5X_2 \leq 180$$

$$X_1, X_2 \geq 0$$

Example 5.2: A company manufactures two products *A* and *B*. These products are processed in the same machine. It takes 10 minutes to process one unit of product *A* and 2 minutes for each unit of product *B* and the machine operates for a maximum of 35 hours in a week. Product *A* requires 1 kg and *B* 0.5 kg of raw material per unit, the supply of which is 600 kg per week. The market constraint on product *B* is known to be 800 units every week. Product *A* costs ` 5 per unit and is sold at ` 10. Product *B* costs ` 6 per unit and can be sold in the market at a unit price of ` 8. Determine the number of units of *A* and *B* that should be manufactured per week to maximize the profit.

NOTES

Solution:

Decision variables: Let X_1 and X_2 be the number of products of *A* and *B*.

Objective function: Cost of product *A* per unit is ` 5 and is sold at ` 10 per unit.

$\therefore$ Profit on one unit of product $A = 10 - 5 = 5$

$\therefore$ X_1 units of product *A*, contributes a profit of ` $5X_1$ from one unit of product.

Similarly, profit on one unit of $B = 8 - 6 = 2$

$\therefore$ X_2 units of product *B*, contribute a profit of ` $2X_2$.

$\therefore$ The objective function is given by,

$$\text{Max } Z = 5X_1 + 2X_2$$

Constraints: Time requirement constraint is given by,

$$10X_1 + 2X_2 \leq (35 \times 60)$$
$$10X_1 + 2X_2 \leq 2100$$

Raw material constraint is given by,

$$X_1 + 0.5X_2 \leq 600$$

Market demand on product *B* is 800 units every week.

$\therefore X_2 \geq 800$

The complete LPP is,

$$\text{Max } Z = 5X_1 + 2X_2$$

Subject to, $10X_1 + 2X_2 \leq 2100$

$$X_1 + 0.5X_2 \leq 600$$
$$X_2 \geq 800$$
$$X_1, X_2 \geq 0$$

Example 5.3: A person requires 10, 12 and 12 units of chemicals *A*, *B* and *C*, respectively for his garden. A liquid product contains 5, 2 and 1 units of *A*, *B* and *C* respectively per jar. A dry product contains 1, 2 and 4 units of *A, B, C* per carton. If the liquid product sells for ` 3 per jar and the dry product sells for ` 2 per carton, what should be the number of jar that needs to be purchased, in order to bring down the cost and meet the requirements?

Solution:

Decision variables: Let X_1 and X_2 be the number of units of liquid and dry products.

Objective function: Since the cost for the products are given, we have to minimize the cost.

Min $Z = 3X_1 + 2X_2$

Constraints: As there are three chemicals and their requirements are given, we have three constraints for these three chemicals.

$$5X_1 + X_2 \geq 10$$
$$2X_1 + 2X_2 \geq 12$$
$$X_1 + 4X_2 \geq 12$$

NOTES

Hence, the complete LPP is,

Min $Z = 3X_1 + 2X_2$

Subject to,

$$5X_1 + X_2 \geq 10$$
$$2X_1 + 2X_2 \geq 12$$
$$X_1 + 4X_2 \geq 12$$
$$X_1, X_2 \geq 0$$

Example 5.4: A paper mill produces two grades of paper, *X* and *Y*. Because of raw material restrictions, it cannot produce more than 400 tonnes of grade *X* and 300 tonnes of grade *Y* in a week. There are 160 production hours in a week. It requires 0.2 and 0.4 hours to produce a tonne of products *X* and *Y* respectively with corresponding profits of ` 200 and ` 500 per tonne. Formulate this as a LPP to maximize profit and find the optimum product mix.

Solution:

Decision variables: Let X_1 and X_2 be the number of units of the two grades of paper, *X* and *Y*.

Objective function: Since the profit for the two grades of paper *X* and *Y* are given, the objective function is to maximize the profit.

Max $Z = 200X_1 + 500X_2$

Constraints: There are two constraints one with reference to raw material, and the other with reference to production hours.

Max $Z = 200X_1 + 500X_2$

Subject to,

$$X_1 \leq 400$$
$$X_2 \leq 300$$
$$0.2X_1 + 0.4X_2 \leq 160$$

Non-negative restriction $X_1, X_2 \geq 0$

Example 5.5: A company manufactures two products *A* and *B*. Each unit of *B* takes twice as long to produce as one unit of *A* and if the company were to produce only *A* it would have time to produce 2000 units per day. The availability of the raw material is enough to produce 1500 units per day of both *A* and *B* together. Product *B* requiring a special ingredient, only 600 units of it can be made per day. If *A* fetches a profit of ` 2 per unit and *B* a profit of ` 4 per unit, find the optimum product mix by graphical method.

Solution: Let X_1 and X_2 be the number of units of the products *A* and *B*, respectively.

NOTES

The profit after selling these two products is given by the objective function,

Max $Z = 2X_1 + 4X_2$

Since the company can produce at the most 2000 units of the product in a day and product *B* requires twice as much time as that of product *A*, production restriction is given by,

$$X_1 + 2X_2 \leq 2000$$

Since the raw material is sufficient to produce 1500 units per day of both *A* and *B*, we have $X_1 + X_2 \leq 1500$.

There are special ingredients for the product *B* we have $X_2 \leq 600$.

Also, since the company cannot produce negative quantities $X_1 \geq 0$ and $X_2 \geq 0$.

Hence, the problem can be finally put in the form:

Find X_1 and X_2 such that the profits, $Z = 2X_1 + 4X_2$ is maximum.

Subject to,

$$X_1 + 2X_2 \leq 2000$$
$$X_1 + X_2 \leq 1500$$
$$X_2 \leq 600$$
$$X_1,\ X_2 \geq 0$$

Example 5.6: A firm manufacturers three products *A*, *B* and *C*. The profits are ` 3, ` 2 and ` 4 respectively. The firm has two machines and the following is the required processing time in minutes for each machine on each product.

		Product		
		A	*B*	*C*
Machines	*C*	4	3	5
	D	3	2	4

Machine *C* and *D* have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 units of *A*, 200 units of *B* and 50 units of *C*, but not more than 150 units of *A*. Set up an LP problem to maximize the profit.

Solution: Let X_1, X_2, X_3 be the number of units of the product *A*, *B*, *C* respectively.

Since the profits are ` 3, ` 2 and ` 4 respectively, the total profit gained by the firm after selling these three products is given by,

$$Z = 3X_1 + 2X_2 + 4X_3$$

The total number of minutes required in producing these three products at machine *C* is given by $4X_1 + 3X_2 + 5X_3$ and at machine *D* is given by,

$3X_1 + 2X_2 + 4X_3$.

The restrictions on the machine *C* and *D* are given by 2000 minutes and 2500 minutes.

$$4X_1 + 3X_2 + 5X_3 \leq 2000$$
$$3X_1 + 2X_2 + 4X_3 \leq 2500$$

NOTES

Also, since the firm manufactures 100 units of A, 200 units of B and 50 units of C, but not more than 150 units of A, the further restriction becomes,

$$100 \leq X_1 \leq 150$$
$$200 \leq X_2 \geq 0$$
$$50 \leq X_3 \geq 0$$

Hence, the allocation problem of the firm can be finally put in the following form:

Find the value of X_1, X_2, X_3 so as to maximize,

$Z = 3X_1 + 2X_2 + 4X_3$

Subject to the constraints,

$$4X_1 + 3X_2 + 5X_3 \leq 2000$$
$$3X_1 + 2X_2 + 4X_3 \leq 2500$$

$$100 \leq X_1 \leq 150,\ 200 \leq X_2 \geq 0, 50 \leq X_3 \geq 0$$

Example 5.7: A peasant has a 100 acres farm. He can sell all potatoes, cabbage or brinjals and can increase the cost to get Re 1.00 per kg for potatoes, Re 0.75 a head for cabbage and ` 2.00 per kg for brinjals. The average yield per acre is 2000 kg of potatoes, 3000 heads of cabbage and 1000 kg of brinjals. Fertilizers can be bought at Re 0.50 per kg and the amount needed per acre is 100 kg each for potatoes and cabbage and 50 kg for brinjals. Manpower required for sowing, cultivating and harvesting per acre is 5 man-days for potatoes and brinjals and 6 man-days for cabbage. A total of 400 man-days of labour is available at ` 20 per man-day. Solve this example as a linear programming model to increase the peasant's profit.

Solution: Let X_1, X_2, X_3 be the area of his farm to grow potatoes, cabbage and brinjals respectively. The peasant produces $2000X_1$ kg of potatoes, $3000X_2$ heads of cabbage and $1000X_3$ kg of brinjals.

$\therefore$ The total sales of the peasant will be,

= ` $(2000X_1 + 0.75 \times 3000X_2 + 2 \times 1000X_3)$

$\therefore$ Fertilizer expenditure will be,

= ` $20\ (5X_1 + 6X_2 + 5X_3)$

$\therefore$ Peasant's profit will be,

Z = Sale (in `) – Total expenditure (in `)

$= (2000X_1 + 0.75 \times 3000X_2 + 2 \times 1000X_3) - 0.5 \times [100(X_1 + X_2) + 50X_2]$

$\quad -20 \times (5X_1 + 6X_2 + 5X_3)$

$Z = 1850X_1 + 2080X_2 + 1875X_3$

Since the total area of the farm is restricted to 100 acres,

$X_1 + X_2 + X_3 \leq 100$

Also, the total man-days manpower is restricted to 400 man-days.

$$5X_1 + 6X_2 + 5X_3 \leq 400$$

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Hence, the peasant's allocation problem can be finally put in the following form:

Find the value of X_1, X_2 and X_3 so as to maximize,

$Z = 1850X_1 + 2080X_2 + 1875X_3$

Subject to,

$$X_1 + X_2 + X_3 \le 100$$
$$5X_1 + 6X_2 + 5X_3 \le 400$$
$$X_1, X_2, X_3 \ge 0$$

Example 5.8: ABC company produces two products: juicers and washing machines. Production happens in two different departments, I and II. Juicers are made in department I and washing machines in department II. These two items are sold weekly. The weekly production should not cross 25 juicers and 35 washing machines. The organization always employs a total of 60 employees in two departments. A juicer requires two man-weeks labour, while a washing machine needs one man-week labour. A juicer makes a profit of ` 60 and a washing machine contributes a profit of ` 40. How many units of juicers and washing machines should the organization make to achieve the maximum profit? Formulate this as an LPP.

Solution: Let X_1 and X_2 be the number of units of juicers and washing machines to be produced.

Each juicer and washing machine contributes a profit of ` 60 and ` 40. Hence, the objective function is to maximize $Z = 60X_1 + 40X_2$.

There are two constraints which are imposed: weekly production and labour.

Since the weekly production cannot exceed 25 juicers and 35 washing machines, therefore

$$X_1 \le 25$$
$$X_2 \le 35$$

A juicer needs two man-weeks of hard works and a washing machine needs one man-week of hard work and the total number of workers is 60.

$$2X_1 + X_2 \le 60$$

Non-negativity restrictions: Since the number of juicers and washing machines produced cannot be negative, we have $X_1 \ge 0$ and $X_2 \ge 0$.

Hence, the production of juicers and washing machines problem can be finally put in the form of a LP model as given below:

Find the value of X_1 and X_2 so as to maximize,

$Z = 60X_1 + 40X_2$

Subject to,

$$X_1 \le 25$$
$$X_2 \le 35$$
$$2X_1 + X_2 \le 60$$
$$\text{and, } X_1, X_2 \ge 0$$

5.5 APPLICATIONS AND LIMITATIONS OF LINEAR PROGRAMMING PROBLEM

The applications of linear programming problems are based on linear programming matrix coefficients and data transmission prior to solving the simplex algorithm. The problem can be formulated from the problem statement using linear programming techniques. The following are the objectives of linear programming:

- Identify the objective of the linear programming problem, i.e., which quantity is to be optimized. For example, maximize the profit.
- Identify the decision variables and constraints used in linear programming, for example, production quantities and production limitations are taken as decision variables and constraints.
- Identify the objective functions and constraints in terms of decision variables using information from the problem statement to determine the proper coefficients.
- Add implicit constraints, such as non-negative restrictions.
- Arrange the system of equations in a consistent form and place all the variables on the left side of the equations.

Applications of Linear Programming

Linear programming problems are associated with the efficient use of allocation of limited resources to meet desired objectives. A solution required to solve the linear programming problem is termed as optimal solution. The linear programming problems contain a very special subclass and depend on mathematical model or description. It is evaluated using relationships and are termed as straight-line or linear. The following are the applications of linear programming:

- Transportation problem
- Diet problem
- Matrix games
- Portfolio optimization
- Crew scheduling

Linear programming problem may be solved using a simplified version of the simplex technique called transportation method. Because of its major application in solving problems involving several product sources and several destinations of products, this type of problem is frequently called the transportation problem. It gets its name from its application to problems involving transporting products from several sources to several destinations. The formation is used to represent more general assignment and scheduling problems as well as transportation and distribution problems. The two common objectives of such problems are as follows:

- To minimize the cost of shipping m units to n destinations.
- To maximize the profit of shipping m units to n destinations.

The goal of the diet problem is to find the cheapest combination of foods that will satisfy all the daily nutritional requirements of a person. The problem is formulated as a linear program where the objective is to minimize cost and meet constraints which require that nutritional needs be satisfied. The constraints are used to regulate the number of calories and amounts of vitamins, minerals, fats, sodium and cholesterol in the diet.

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Game method is used to turn a matrix game into a linear programming problem. It is based on the Min-Max theorem which suggests that each player determines the choice of strategies on the basis of a probability distribution over the player's list of strategies.

The portfolio optimization template calculates the optimal capital of investments that gives the highest return for the least risk. The unique design of the portfolio optimization technique helps in financial investments or business portfolios. The optimization analysis is applied to a portfolio of businesses to represent a desired and beneficial framework for driving capital allocation, investment and divestment decisions.

Crew scheduling is an important application of linear programming problem. It helps if any airline has a problem related to a large potential crew schedules variables. Crew scheduling models are a key to airline competitive cost advantage these days because crew costs are the second largest flying cost after fuel costs.

Limitations of Linear Programming Problems

Linear programming is applicable if constraints and objective functions are linear, but there are some limitations of this technique which are as follows:

- All the uncertain factors, such as weather conditions, growth rate of industry, etc., are not taken into consideration.
- Integer values are not taken as the solution, e.g., a value is required for fraction and the nearest integer is not taken for the optimal solution.
- Linear programming technique gives those practical-valued answers that are really not desirable with respect to linear programming problem.
- It deals with one single objective in real life problem which is more limited and the problems come with multi-objective.
- In linear programming, coefficients and parameters are assumed as constants but in realty they do not take place.
- Blending is a frequently encountered problem in linear programming. For example, if different commodities are purchased which have different characteristics and costs, then the problem helps to decide how much of each commodity would be purchased and blended within specified bound so that the total purchase cost is minimized.

5.6 SOLUTION OF LINEAR PROGRAMMING PROBLEM

The linear programming problems can be solved as follows:

5.6.1 Graphical Solution

Simple linear programming problem with two decision variables can be easily solved by graphical method.

Procedure for Solving LPP by Graphical Method

The steps involved in the graphical method are as follows:

Step 1: Consider each inequality constraint as an equation.

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Step 2: Plot each equation on the graph as each will geometrically represent a straight line.

Step 3: Mark the region. If the inequality constraint corresponding to that line is $\leq$, then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint $\geq$ sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region, thus obtained, is called the feasible region.

Step 4: Allocate an arbitrary value, say zero, for the objective function.

Step 5: Draw the straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

Step 6: Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passes through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passes through at least one corner of the feasible region.

Step 7: Find the coordinates of the extreme points selected in Step 6 and find the maximum or minimum value of Z.

Note: As the **optimal values** occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one which gives the optimal solution, i.e., in the case of maximization problem, optimal point corresponds to the corner point at which the objective function has a maximum value and in the case of minimization, the corner point which gives the objective function the minimum value is the optimal solution.

Example 5.9: Solve the following LPP by graphical method.

Minimize $Z = 20X_1 + 10X_2$

Subject to,

$$X_1 + 2X_2 \leq 40$$
$$3X_1 + X_2 \geq 30$$
$$4X_1 + 3X_2 \geq 60$$
$$X_1, X_2 \geq 0$$

Solution: Replace all the inequalities of the constraints by equation,

$X_1 + 2X_2 = 40$ If $X_1 = 0 \Rightarrow X_2 = 20$

If $X_2 = 0 \Rightarrow X_1 = 40$

$\therefore X_1 + 2X_2 = 40$ passes through (0, 20) (40, 0)

$3X_1 + X_2 = 30$ passes through (0, 30) (10, 0)

$4X_1 + 3X_2 = 60$ passes through (0, 20) (15, 0)

Plot each equation on the graph.

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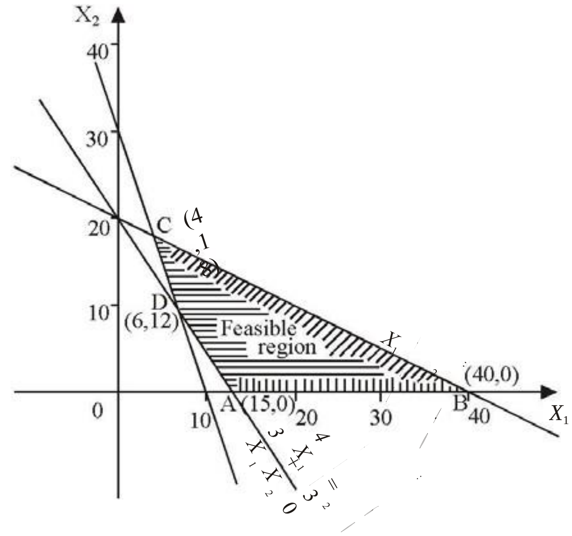


The feasible region is *ABCD*.

C and *D* are points of intersection of lines.

$X_1 + 2X_2 = 40$, $3X_1 + X_2 = 30$

And, $4X_1 + 3X_2 = 60$

On solving, we get *C* (4, 18) and *D* (6, 12)

Corner Points	*Value of* $Z = 20X_1 + 10X_2$
A (15, 0)	300
B (40, 0)	800
C (4, 18)	260
D (6, 12)	240 (Minimum value)

∴ The minimum value of *Z* occurs at *D* (6, 12). Hence, the optimal solution is $X_1 = 6$, $X_2 = 12$.

Example 5.10: Find the maximum value of $Z = 5X_1 + 7X_2$

Subject to the constraints,

$X_1 + X_2 \leq 4$

$3X_1 + 8X_2 \leq 24$

$10X_1 + 7X_2 \leq 35$

$X_1, X_2 > 0$

Solution: Replace all the inequalities of the constraints by forming equations.

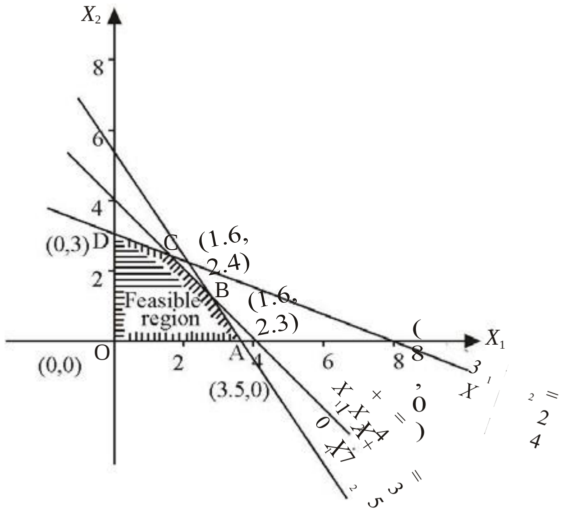
$X_1 + X_2 = 4$ passes through (0, 4) (4, 0)

$3X_1 + 8X_2 = 24$ passes through (0, 3) (8, 0)

$10X_1 + 7X_2 = 35$ passes through (0, 5) (3.5, 0)

Plot these lines in the graph and mark the region below the line as the inequality of the constraint is ≤ and is also lying in the first quadrant.

NOTES



The feasible region is *OABCD.*

B and *C* are points of instruction of lines,

$X_1 + X_2 = 4$, $10X_1 + 7X_2 = 35$

And, $3X_1 + 8X_2 = 24$,

On solving we get,

B (1.6, 2.3)

C (1.6, 2.4)

Corner Points	*Value of* $Z = 5X_1 + 7X_2$
O (0, 0)	0
A (3.5, 0)	17.5
B (1.6, 2.3)	25.1
C (1.6, 2.4)	24.8 (Maximum value)
D (0, 3)	21

$\therefore$ The maximum value of *Z* occurs at *C* (1.6, 2.4) and the optimal solution is $X_1 = 1.6, X_2 = 2.4$.

Example 5.11: A company makes 2 types of hats. Each hat *A* needs twice as much labour time as the second hat *B*. If the company is able to produce only hat *B*, then it can make about 500 hats per day. The market limits daily sales of the hat *A* and hat *B* to 150 and 250 hats. The profits on hat *A* and hat *B* are ` 8 and ` 5, respectively. Solve graphically to get the optimal solution.

Solution: Let X_1 and X_2 be the number of units of type *A* and type *B* hats respectively.

Maximize $Z = 8X_1 + 5X_2$

Subject to, $2X_1 + 2X_2 \leq 500$

$X_1 \geq 150$

$X_2 \geq 250$

$X_1, X_2 \geq 0$

NOTES

First rewrite the inequality of the constraint into an equation and plot the lines in the graph.

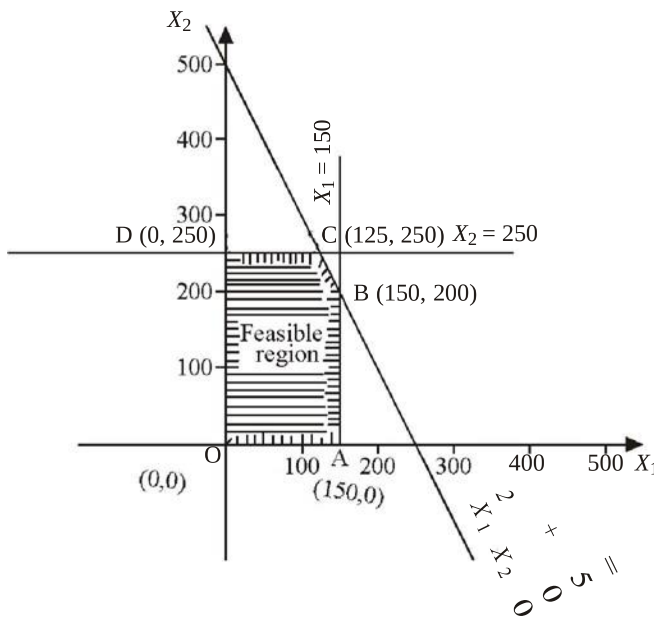
$$2X_1 + X_2 = 500 \text{ passes through } (0, 500)\ (250, 0)$$
$$X_1 = 150 \text{ passes through } (150, 0)$$
$$X_2 = 250 \text{ passes through } (0, 250)$$

We mark the region below the lines lying in the first quadrant as the inequality of the constraints are ≤. The feasible region is *OABCD*. *B* and *C* are points of intersection of lines:

$2X_1 + X_2 = 500$, where $X_1 = 150$ and $X_2 = 250$

On solving, we get *B* (150, 200)

C (125, 250)



Corner Points	*Value of Z* = $8X_1 + 5X_2$
O (0, 0)	0
A (150, 0)	1200
B (150, 200)	2200
C (125, 250)	2250 (Maximum *Z* = 2250)
D (0, 250)	1250

The maximum value of *Z* is attained at *C* (125, 250)

∴ The optimal solution is $X_1 = 125$, $X_2 = 250$

Therefore, the company should produce 125 hats of type *A* and 250 hats of type *B* in order to get the maximum profit of ` 2250.

Example 5.12: By graphical method solve the following LPP:

Maximize $Z = 3X_1 + 4X_2$

Subject to, $5X_1 + 4X_2 \leq 200$

$3X_1 + 5X_2 \leq 150$

$5X_1 + 4X_2 \geq 100$

$8X_1 + 4X_2 \geq 80$

and $X_1, X_2 \geq 0$

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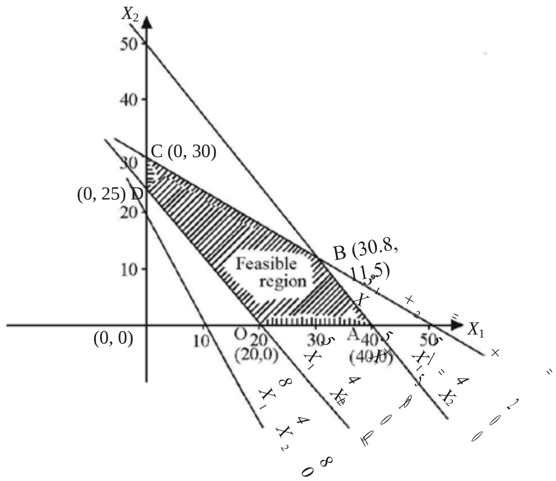
Solution:



Feasible region is given by *OABCD*.

Corner Points	*Value of* $Z = 3X_1 + 4X_2$
O (20, 0)	60
A (40, 0)	120
B (30.8, 11.5)	138.4 (Maximum value)
C (0, 30)	120
D (0, 25)	100

$\therefore$ The maximum value of Z is attained at B (30.8, 11.5)

$\therefore$ The optimal solution is $X_1 = 30.8, X_2 = 11.5$

Example 5.13: Use graphical method to solve the following LPP:

Maximize, $Z = 6X_1 + 4X_2$

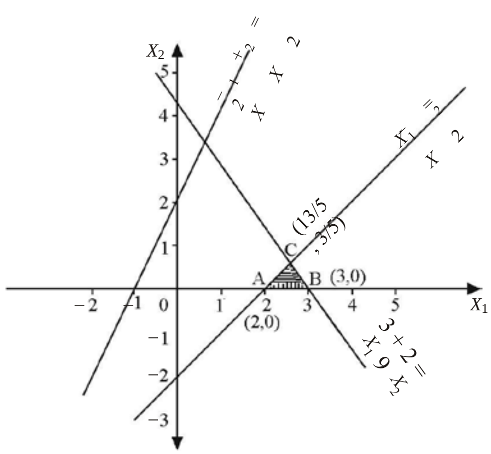
Subject to, $-2X_1 + X_2 \leq 2$

$X_1 - X_2 \leq 2$

$3X_1 + 2X_2 \leq 9$

$X_1, X_2 \geq 0$

Solution:



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The feasible region is given by *ABC*.

Corner Points	*Value of* $Z = 6X_1 + 4X_2$
A (2, 0)	12
B (3,0)	18
C (13/5, 3/5)	$\frac{90}{5} = 18$ (Maximum value)

The maximum value of *Z* is attained at *C* (13/5, 3/5)

$\therefore$ The optimal solution is $X_1 = 13/5$, $X_2 = 3/5$

Example 5.14: Use graphical method to solve the following LPP.

Minimize, $Z = 3X_1 + 2X_2$

Subject to, $5X_1 + X_2 \geq 10$

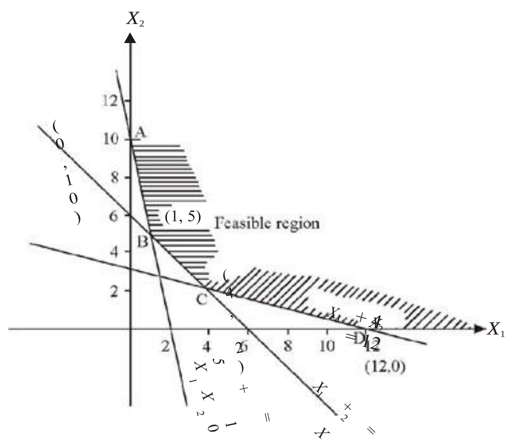
$X_1 + X_2 \geq 6$

$X_1 + 4X_2 \geq 12$

$X_1, X_2 \geq 0$

Solution:

Corner Points	*Value of* $Z = 3X_1 + 2X_2$
A (0, 10)	20
B (1, 5)	13 (Minimum value)
C (4, 2)	16
D (12, 0)	36



Since the minimum value is attained at *B* (1, 5), the optimum solution is,

$X_1 = 1$, $X_2 = 5$

Note: In this problem, if the objective function is maximization then the solution is unbounded, as maximum value of *Z* occurs at infinity.

Some More Cases

There are some linear programming problems which may have:

(i) A unique optimal solution

(ii) An infinite number of optimal solutions

(iii) An unbounded solution

(iv) No solution

Following examples will illustrate these cases.

Example 5.15: Solve the following LPP by graphical method.

Maximize $Z = 100X_1 + 40X_2$

Subject to, $5X_1 + 2X_2 \leq 1000$

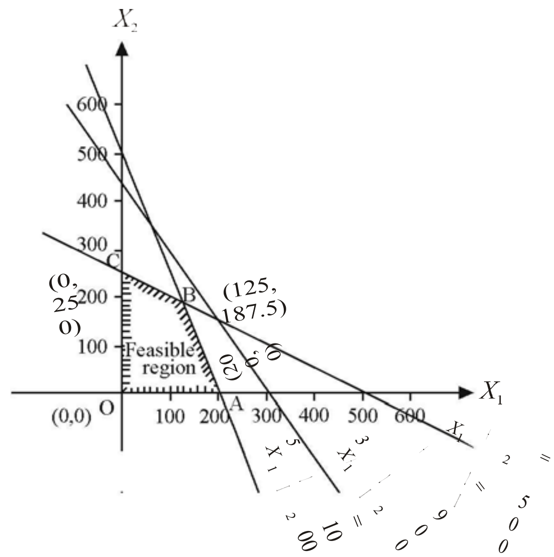
$3X_1 + 2X_2 \leq 900$

$X_1 + 2X_2 \leq 500$

and $X_1 + X_2 \geq 0$

NOTES

Solution:



The solution space is given by the feasible region *OABC*.

Corner Points	*Value of* $Z = 100X_1 + 40X_2$
O (0, 0)	0
A (200, 0)	20,000
B (125, 187.5)	20,000 (Maximum value of *Z*)
C (0, 250)	10,000

$\therefore$ The maximum value of *Z* occurs at two vertices *A* and *B*.

Since there are infinite number of points on the line, joining *A* and *B* gives the same maximum value of *Z*.

Thus, there are infinite number of optimal solutions for the LPP.

Example 5.16: Solve the following LPP.

Maximize $Z = 3X_1 + 2X_2$

Subject to, $X_1 + X_2 \geq 1$

$X_1 + X_2 \geq 3$

$X_1, X_2 \geq 0$

Solution: The solution space is unbounded. The value of the objective function at the vertices *A* and *B* are *Z* (*A*) = 6, *Z* (*B*) = 6. But, there exist points in the convex region for which the value of the objective function is more than 8. In fact, the maximum value of *Z* occurs at infinity. Hence, the problem has an unbounded solution.

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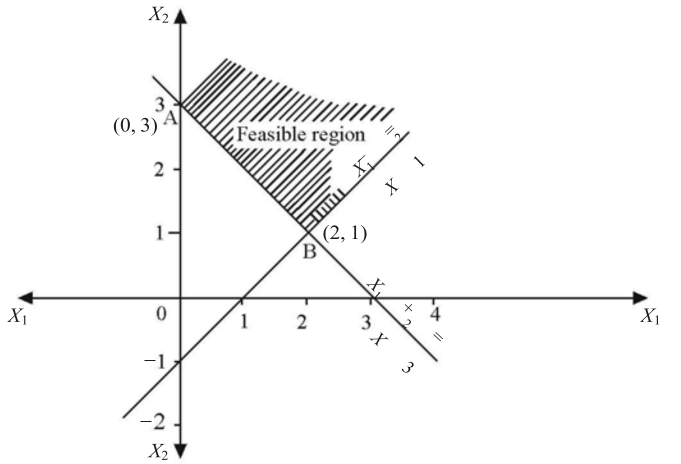
No feasible solution.



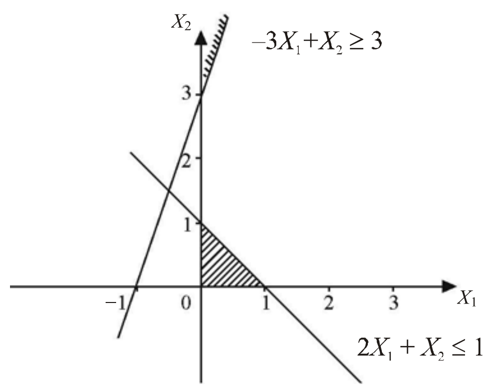
When there is no feasible region formed by the constraints in conjunction with non-negativity conditions, no solution to the LPP exists.

Example 5.17: Solve the following LPP.

Maximize $Z = X_1 + X_2$

Subject to the constraints,

$$X_1 + X_2 \le 1$$
$$-3X_1 + X_2 \ge 3$$
$$X_1, X_2 \ge 0$$



Solution: There being no point (X_1, X_2) common to both the shaded regions, we cannot find a feasible region for this problem. So, the problem cannot be solved. Hence, the problem has no solution.

5.6.2 Some Important Definitions

The following are some of the important definitions:

1. A set of values $X_1, X_2 \dots X_n$ which satisfies the constraints of the LPP is called its solution.
2. Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its feasible solution.
3. Any feasible solution which optimizes (minimizes or maximizes) the objective function of the LPP is called its optimum solution.

4. Given a system of m linear equations with n variables ($m < n$), any solution which is obtained by solving m variables keeping the remaining $n - m$ variables zero is called a basic solution. Such m variables are called basic variables and the remaining variables are called non-basic variables.

NOTES

5. A basic feasible solution is a basic solution which also satisfies all basic variables are non-negative.

   Basic **feasible solutions** are of following two types:

   (i) ***Non-degenerate:*** A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive X_i ($i = 1, 2, ..., m$), i.e., none of the basic variables is zero.

   (ii) ***Degenerate:*** A basic feasible solution is said to be degenerate if one or more basic variables are zero.

6. If the value of the objective function Z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

5.6.3 Canonical or Standard Forms of LPP

The general LPP can be put in either canonical or standard forms.

In the standard form, irrespective of the objective function, namely maximize or minimize, all the constraints are expressed as equations. Moreover, RHS of each constraint and all variables are non-negative.

Characteristics of the Standard Form

The following are the characteristics of the standard form:

(i) The objective function is of maximization type.

(ii) All constraints are expressed as equations.

(iii) Right hand side of each constraint is non-negative.

(iv) All variables are non-negative.

In the canonical form, if the objective function is of maximization, all the constraints other than non-negativity conditions are '≤' type. If the objective function is of minimization, all the constraints other than non-negative conditions are '≥' type.

Characteristics of the Canonical Form

The following are the characteristics of the canonical form:

(i) The objective function is of maximization type.

(ii) All constraints are of '≤' type.

(iii) All variables X_i are non-negative.

Notes:

1. Minimization of a function Z is equivalent to maximization of the negative expression of this function, i.e., Min $Z = –$Max $(–Z)$.
2. An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by (–1).
3. Suppose we have the constraint equation,

$$a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n = b_1$$

   This equation can be replaced by two weak inequalities in opposite directions.

$$a_{11}X_1 + a_{12}X_2 + ... + a_m X_n \le b_1$$
$$a_{11}X_1 + a_{12}X_2 + ... + a_{1n} X_n \ge b_1$$

NOTES

4. If a variable is unrestricted in sign, then it can be expressed as a difference of two non-negative variables, i.e., X_1 is unrestricted in sign, then $X_i = X_i' - X_i''$, where X_i, X_i', X_i'' are ≥ 0.
5. In standard form, all the constraints are expressed in equation, which is possible by introducing some additional variables called **slack variables** and **surplus variables** so that a system of simultaneous linear equations is obtained. The necessary transformation will be made to ensure that $b_i \ge 0$.

Definition of Slack and Surplus Variables

(i) If the constraints of a general LPP be,

$$\sum_{j=1}^{n} a_{ij} \ X_i \le b_i \ (i = 1, 2, ..., m),$$

then the non-negative variables S_i, which are introduced to convert the inequalities (≤) to the equalities $\sum_{j=1}^{n} a_{ij} \ X_i + S_i = b_i \ (i = 1, 2, ..., m)$, are called slack variables.

Slack variables are also defined as the non-negative variables which are added in the LHS of the constraint to convert the inequality '≤' into an equation.

(ii) If the constraints of a general LPP be,

$$\sum_{j=1}^{n} a_{ij} \ X_j \ge b_i \ (i = 1, 2, ..., m),$$

then, the non-negative variables S_i which are introduced to convert the inequalities (≥) to the equalities $\sum_{j=1}^{n} a_{ij} \ X_j - S_i = b_i \ (i = 1, 2, ..., m)$ are called surplus variables.

Surplus variables are defined as the non-negative variables which are removed from the LHS of the constraint to convert the inequality '≥' into an equation.

5.6.4 Simplex Method

Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be that at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solution.

Definition

(i) Let X_B be a basic feasible solution to the LPP.

Max $Z = C_X$

Subject to $A_X = b$ and $X \ge 0$, such that it satisfies $X_B = B^{-1}b$,

Where B is the basic matrix formed by the column of basic variables.

The vector $C_B = (C_{B1}, C_{B2} \dots C_{Bm})$, where C_{Bj} are components of C associated with the basic variables is called the cost vector associated with the basic feasible solution X_B.

(ii) Let X_B be a basic feasible solution to the LPP.

NOTES

Max $Z = C_X$, where $A_X = b$ and $X \geq 0$.

Let C_B be the cost vector corresponding to X_B. For each column vector a_j in A_1, which is not a column vector of B, let

$$a_j = \sum_{i=1}^{m} a_{ij} b_j$$

Then the number $Z_j = \sum_{i=1}^{m} C_{Bi} a_{ij}$ is called the evaluation corresponding to a_j and the number $(Z_j - C_j)$ is called the net evaluation corresponding to j.

Simplex Algorithm

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows:

Step 1: Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximization by,

Min $Z = -\text{Max}(-Z)$

Step 2: Check whether all b_i ($i = 1, 2, \ldots, m$) are positive. If any one of b_i is negative, then multiply the inequation of the constraint by –1 so as to get all b_i to be positive.

Step 3: Express the problem in the standard form by introducing slack/surplus variables to convert the inequality constraints into equations.

Step 4: Obtain an initial basic feasible solution to the problem in the form $X_B = B^{-1}b$ and put it in the first column of the simplex table. Form the initial simplex table shown as follows:

		C_j	C_1	C_2	C_3	..	..	0 0 0
C_B	S_B	X_B	X_1	X_2	X_3	X_4	..... X_n	S_1.. S_2 ..... S_m
C_{B1}	S_1	b_1	a_{11}	a_{12}	a_{13}	a_{14}	..... a_{1n}	1 0 0
C_{B2}	S_2	b_2	a_{21}	a_{22}	a_{23}	a_{24}	..... a_{2n}	1 0 0

Step 5: Compute the net evaluations $Z_j - C_j$ by using the relation:

$Z_j - C_j = C_B (a_j - C_j)$

Examine the sign of $Z_j - C_j$:

(i) If all $Z_j - C_j \geq 0$, then the initial basic feasible solution X_B is an optimum basic feasible solution.

(ii) If at least one $Z_j - C_j > 0$, then proceed to the next step as the solution is not optimal.

Step 6: To find the entering variable, i.e., key column.

If there are more than one negative $Z_j - C_j$ choose the most negative of them. Let it be $Z_r - C_r$ for some $j = r$. This gives the entering variable X_r and is indicated by an arrow at the bottom of the rth column. If there are more than one variable having the

NOTES

same most negative $Z_j - C_j$, then any one of the variable can be selected arbitrarily as the entering variable.

(i) If all $X_{ir} \leq 0$ $(i = 1, 2, \ldots, m)$ then there is an unbounded solution to the given problem.

(ii) If at least one $X_{ir} > 0$ $(i = 1, 2, \ldots, m)$, then the corresponding vector X_r enters the basis.

Step 7: To find the leaving variable or key row:

Compute the ratio $(X_{Bi}/X_{kr}, X_{ir}>0)$

If the minimum of these ratios be X_{Bi}/X_{kr}, then choose the variable X_k to leave the basis called the key row and the element at the intersection of the key row and the key column is called the key element.

Step 8: Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under C_B column. The leaving element is converted to unity by dividing the key equation by the key element and all other elements in its column to zero by using the formula:

$$\text{New element} = \text{Old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{Key element}}\right]$$

Step 9: Repeat the procedure of Step (5) until either an optimum solution is obtained or there is an indication of unbounded solution.

Example 5.18: Use simplex method to solve the following LPP:

Maximize $Z = 3X_1 + 2X_2$

Subject to, $X_1 + X_2 \leq 4$

$X_1 - X_2 \leq 2$

$X_1, X_2 \geq 0$

Solution: By introducing the slack variables S_1, S_2, convert the problem into standard form.

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2$

Subject to, $X_1 + X_2 + S_1 = 4$

$X_1 - X_2 + S_2 = 2$

$X_1, X_2, S_1, S_2 \geq 0$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

An initial basic feasible solution is given by,

$X_B = B^{-1}b$,

Where, $B = I_2$, $X_B = (S_1, S_2)$

i.e., $(S_1, S_2) = I_2\,(4, 2) = (4, 2)$

Initial Simplex Table

$Z_j = C_B a_j$

$Z_1 - C_1 = C_B a_1 - C_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1\ 1) - 3 = -3$

$Z_2 - C_2 = C_B a_2 - C_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1\ 1) - 2 = -2$

$Z_3 - C_3 = C_B a_3 - C_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1\ 0) - 0 = -0$

$Z_4 - C_4 = C_B a_4 - C_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (0\ 1) - 0 = -0$

		C_j	3	2	0	0	
C_B	B	X_B	X_1	X_2	S_1	S_2	Min $\frac{X_B}{X_1}$
0	S_1	4	1	1	1	0	4/1 = 4
←0	S_2	2	(1)	–1	0	1	2/1 = 2
	Z_j	0	0	0	0	0	
	$Z_j - C_j$		–3↑	–2	0	0	

Since, there are some $Z_j - C_j = 0$, the current basic feasible solution is not optimum.

Since, $Z_1 - C_1 = -3$ is the most negative, the corresponding non-basic variable X_1 enters the basis.

The column corresponding to this X_1 is called the key column.

$$\text{Ratio} = \text{Min}\left\{\frac{X_{Bi}}{X_{ir}}, X_{ir} > 0\right\}$$

$$= \text{Min}\left\{\frac{4}{1}, \frac{2}{1}\right\}, \text{ which corresponds to } S_2$$

∴ The leaving variable is the basic variable S_2. This row is called the key row. Convert the leading element X_{21} to units and all other elements in its column n, i.e., (X_1) to zero by using the formula:

New element = Old element –

$$\left[\frac{\text{Product of elements in key row and key column}}{\text{Key element}}\right]$$

To apply this formula, first we find the ratio, namely

$$\frac{\text{The element to be zero}}{\text{Key element}} = \frac{1}{1} = 1$$

Apply this ratio for the number of elements that are converted in the key row. Multiply this ratio by key row element shown as follows:

1×2

1×1

1×-1

1×0

1×1

NOTES

Now, subtract this element from the old element. The element to be converted into zero is called the old element row. Finally, we have

$4 - 1 \times 2 = 2$

$1 - 1 \times 1 = 0$

$1 - 1 \times -1 = 2$

$1 - 1 \times 0 = 1$

$0 - 1 \times 1 = -1$

$\therefore$ The improved basic feasible solution is given in the following simplex table:

First Iteration

		C_j	3	2	0	0	
C_B	B	X_B	X_1	X_2	S_1	S_2	Min $\frac{X_B}{X_2}$
←0	S_1	2	0	(2)	1	–1	2/2 = 1
3	X_1	2	1	–1	0	1	–
	Z_j	6	3	–3	0	0	
	$Z_j - C_j$		0	–5↑	0	0	

Since, $Z_2 - C_2$ is the most negative, X_2 enters the basis.

To find Min $\left(\frac{X_B}{X_{i2}}, X_{i2} > 0 \right)$

Min $\left(\frac{2}{2} \right)$

This gives the outgoing variables. Convert the leaving element into one. This is done by dividing all the elements in the key row by 2. The remaining elements are converted to zero by using the following formula.

Here, $-\,{}^{1}\!/_{2}$ is the common ratio. Put this ratio 5 times and multiply each ratio by the key row element.

$-\frac{1}{2} \times 2$

$-\frac{1}{2} \times 0$

$-\frac{1}{2} \times 2$

$-1/2 \times 1$

$-1/2 \times -1$

Subtract this from the old element. All the row elements which are converted into zero are called the old elements.

NOTES

$$2-\left(-\frac{1}{2}\times 2\right)=3$$

1 – (–1/2 × 0) = 1

–1 – (–1/2 × 2) = 0

0 – (–1/2 × 1) = 1/2

1 – (–1/2 × –1) = 1/2

Second Iteration

		C_j	3	2	0	0
C_B	B	X_B	X_1	X_2	S_1	S_2
2	X_2	1	0	1	1/2	–1/2
3	X_1	3	1	0	1/2	1/2
	Z_j	11	3	2	5/2	1/2
	$Z_j–C_j$		0	0	5/2	1/2

Since all $Z_j - C_j \geq 0$, the solution is optimum. The optimal solution is Max $Z = 11$, $X_1 = 3$, and $X_2 = 1$.

Example 5.19: Solve the LPP when,

Maximize $Z = 3X_1 + 2X_2$

Subject to, $4X_1 + 3X_2 \leq 12$

$4X_1 + X_2 \leq 8$

$4X_1 - X_2 \leq 8$

$X_1, X_2 \geq 0$

Solution: Convert the inequality of the constraint into an equation by adding slack variables S_1, S_2, S_3.

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 + 0S_3$

Subject to, $4X_1 + 3X_2 + S_1 = 12$

$4X_1 + X_2 + S_2 = 8$

$4X_1 - X_2 + S_3 = 8$

$X_1, X_2, S_1, S_2, S_3 \geq 0$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 & S_3 \\ 4 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 8 \end{bmatrix}$$

NOTES

Initial Table

C_B	B	X_B	X_1 (3)	X_2 (2)	S_1 (0)	S_2 (0)	S_3 (0)	Min $\frac{X_B}{X_1}$
0	S_1	12	4	3	1	0	0	12/4 = 3
0	S_2	8	4	1	0	1	0	8/4 = 2
←0	S_3	8	(4)	–1	0	0	1	8/4 = 2
	Z_j	0	0	0	0	0	0	
	$Z_j – C_j$		–3↑	–2	0	0	0	

$\therefore Z_1 - C_1$ is most negative, X_1 enters the basis and the Min $\left(\frac{X_B}{X_{i1}}, X_{i1} > 0\right)$

= Min (3, 2, 2) gives S_3 as the leaving variable.

Convert the leaving element into 1, by dividing the key row elements by 4 and the remaining elements into 0.

First Iteration

C_B	B	X_B	X_1 (3)	X_2 (2)	S_1 (0)	S_2 (0)	S_3 (0)	Min $\frac{X_B}{X_2}$
0	S_1	4	0	4	1	0	–1	4/4 = 1
←0	S_2	0	0	(2)	0	1	–1	0/2 = 0
3	X_1	2	1	–1/4	0	0	1/4	–
	Z_j	(6)	3	–3/4	0	0	3/4	
	$Z_j – C_j$		0	–11/4↑	0	0	3/4	

$8-\frac{4}{4}\times 8=0 \qquad 12-\frac{4}{4}\times 8=4$

$4-\frac{4}{4}\times 4=0 \qquad 4-\frac{4}{4}\times 4=0$

$1-\frac{4}{4}\times -1=2 \qquad 3-\frac{4}{4}\times -1=4$

$0-\frac{4}{4}\times 0=0 \qquad 1-\frac{4}{4}\times 0=1$

$1-\frac{4}{4}\times 0=1 \qquad 0-\frac{4}{4}\times 0=0$

$0-\frac{4}{4}\times 1=-1 \qquad 0-\frac{4}{4}\times 1=-1$

NOTES

Since, $Z_2 - C_2 = -\frac{3}{4}$ is the most negative, X_2 enters the basis.

To find the outgoing variable, find Min $\left(\frac{X_B}{X_{i2}}, X_{i2} > 0 \right)$.

$$\text{Min}\left(\frac{4}{4}, \frac{0}{2}, -1 \right)$$

First Iteration

Therefore, S_2 leaves the basis. Convert the leaving element into 1 by dividing the key row elements by 2 and the remaining elements in that column into zero using the formula,

New element = Old element

$$-\left[\frac{\text{Product of elements in key row and key column}}{\text{Key element}} \right]$$

		C_j	3	2	0	0	0	
C_B	B	X_B	X_1	X_2	S_1	S_2	S_3	Min $\frac{X_B}{S_3}$
←0	S_1	4	0	0	1	–2	①	4/1 = 4
2	X_2	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	–
3	X_1	2	1	0	0	1/8	1/8	2/1/8 = 16
	Z_i	6	3	2	0	11/8	–5/8	
	$Z_j - C_j$		0	0	0	11/8	–5/8↑	

Second Iteration

Since $Z_5 - C_5$ = –5/8 is the most negative, S_3 enters the basis and,

$$\text{Min}\left(\frac{X_B}{S_{i3}}, S_{i3} \right) = \text{Min}\left(\frac{4}{1}, \frac{2}{1/18} \right)$$

Therefore, S_1 leaves the basis. Convert the leaving element into one and the remaining elements into zero.

Third Iteration

		C_j	3	2	0	0	0
C_B	B	X_B	X_1	X_2	S_1	S_2	S_3
0	S_3	4	0	0	1	–2	1
2	X_2	2	0	1	1/2	–1/2	0
3	X_1	3/2	1	0	–1/8	3/8	0
	Z_j	17/2	3	2	5/8	1/8	0
	$Z_j - C_j$		0	0	5/8	1/8	0

Since all $Z_j - C_j \geq 0$, the solution is optimum and it is given by $X_1 = 3/2$, $X_2 = 2$ and Max $Z = 17/2$.

NOTES

Example 5.20: Using simplex method solve the following LPP.

Maximize $Z = X_1 + X_2 + 3X_3$

Subject to, $3X_1 + 2X_2 + X_3 \leq 3$

$2X_1 + X_2 + 2X_3 \leq 2$

$X_1, X_2, X_3 \geq 0$

Solution: Rewrite the inequality of the constraints into an equation by adding slack variables.

Max $Z = X_1 + X_2 + 3X_3 + 0S_1 + 0S_2$

Subject to, $3X_1 + 2X_2 + X_3 + S_1 = 3$

$2X_1 + X_2 + 2X_3 + S_2 = 2$

Initial basic feasible solution is,

$X_1 = X_2 = X_3 = 0$

$S_1 = 3, S_2 = 2$ and $Z = 0$

$$\begin{bmatrix} X_1 & X_2 & X_3 & S_1 & S_2 \\ 3 & 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & 0 \end{bmatrix}$$

		C_j	3	2	0	0	0	
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2	Min $\frac{X_B}{X_3}$
0	S_1	3	3	2	1	1	0	3/1 = 3
←0	S_2	2	2	1	(2)	0	1	2/2 = 1
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$		–1	–1	–3↑	0	0	

Since $Z_3 - C_3 = -3$ is the most negative, the variable X_3 enters the basis. The column corresponding to X_3 is called the key column.

To determine the key row or leaving variable, find Min $\left(\frac{X_B}{X_3}, X^3 > 0\right)$

Min $\left(\frac{3}{1} = 3, \frac{2}{2} = 1\right)$

Therefore, the leaving variable is the basic variable S_2, the row is called the key row and the intersection element 2 is called the key element.

Convert this element into one by dividing each element in the key row by 2 and the remaining elements in that key column as zero using the formula,

NOTES

New element = Old element

$$-\left[\frac{\text{Product of elements in key row and key column}}{\text{Key element}}\right]$$

First Iteration

		C_j	1	1	3	0	0
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2
0	S_1	2	2	3/2	0	1	–1/2
3	X_3	1	1	1/2	1	0	1/2
	Z_j	3	3	3/2	3	0	3/2
	$Z_j – C_j$		2	1/2	0	0	3/2

Since all $Z_j - C_j \geq 0$, the solution is optimum and it is given by $X_1 = 0$, $X_2 = 0$, $X_3 = 1$, Max $Z = 3$.

Example 5.21: Use simplex method to solve the following LPP.

Minimize $Z = X_2 - 3X_3 + 2X_5$

Subject to,
$$3X_2 - X_3 + 2X_5 \leq 7$$
$$-2X_2 + 4X_3 \leq 12$$
$$-4X_2 + 3X_3 + 8X_5 \leq 10$$
$$X_2, X_3, X_5 \geq 0$$

Solution: Since the given objective function is of minimization we shall convert it into maximization using Min $Z = –\text{Max}(–Z)$.

Max $Z = –X_2 + 3X_3 – 2X_5$

Subject to,
$$3X_2 - X_3 + 2X_5 \leq 7$$
$$-2X_2 + 4X_3 \leq 12$$
$$-4X_2 + 3X_3 + 8X_5 \leq 10$$

We rewrite the inequality of the constraints into an equation by adding slack variables S_1, S_2, S_3 and the standard form of LPP becomes,

Max $Z = –X_2 + 3X_3 – 2X_5 + 0S_1 + 0S_2 + 0S_3$

Subject to,
$$3X_2 - X_3 + 2X_5 + S_1 = 7$$
$$-2X_2 + 4X_3 + S_2 = 12$$
$$-4X_2 + 3X_3 + 8X_5 + S_3 = 10$$
$$X_2, X_3, X_5, S_1, S_2, S_3 \geq 0$$

$\therefore$ The initial basic feasible solution is given by,

S_1=7, S_2=12, S_3=10. (X_2=X_3=X_5=0)

NOTES

Initial Table

		C_j	–1	3	–2	0	0	0	
C_B	B	X_B	X_2	X_3	X_5	S_1	S_2	S_3	Min $\frac{X_B}{X_3}$
0	S_1	7	3	–1	2	1	0	0	–
←0	S_2	12	–2	(4)	0	0	1	0	12/4=3
0	S_3	10	–4	3	8	0	0	1	10/3=3.33
	Z_j	0	0	0	0	0	0	0	
	$Z_j – C_j$		1	–3↑	2	0	0	0	

Since, $Z_2 - C_2 = -3 < 0$, the solution is not optimum.

The incoming variable is X_3 (key column) and the outgoing variable (key row) is given by,

$$\text{Min}\left(\frac{X_B}{X_{i3}}, X_{i3} > 0\right) = \text{Min}\left(\frac{12}{4}, \frac{10}{3}\right)$$

Hence, S_2 leaves the basis.

First Iteration

		C_j	–1	3	–2	0	0	0	
C_B	B	X_B	X_2	X_3	X_5	S_1	S_2	S_3	Min $\frac{X_B}{X_2}$
←0	S_1	10	5/2	0	2	1	1/4	0	10/5/2=4
3	X_3	3	–1/2	1	0	0	1/4	0	–
0	S_3	1	–5/2	0	8	0	–3/4	1	–
	Z_j	9	–3/2	3	0	0	3/4	0	
	$Z_j – C_j$		–1/2↑	0	2	0	3/4	0	

Since $Z_1 - C_1 < 0$, the solution is not optimum. Improve the solution by allowing the variable X_2 to enter into the basis and the variable S_1 to leave the basis.

Second Iteration

		C_j	–1	3	–2	0	0	0
C_B	B	X_B	X_2	X_3	X_5	S_1	S_2	S_3
–1	X_2	4	1	0	4/5	2/5	1/10	0
3	X_3	5	0	1	2/5	1/5	3/10	0
0	S_3	11	0	0	10	1	–1/2	1
	Z_j	11	–1	3	2/5	1/5	8/10	0
	$Z_j–C_j$		0	0	12/5	1/5	8/10	0

Since, $Z_j - C_j \geq 0$, the solution is optimum.

$\therefore$ The optimal solution is given by Max $Z = 11$

$X_2 = 4, X_3 = 5, X_5 = 0$

$\therefore$ Min $Z = -\text{Max}\,(-Z) = -11$

$\therefore$ Min $Z = -11, X_2 = 4, X_3 = 5, X_5 = 0$

Example 5.22: Solve the following LPP using simplex method.

NOTES

$$\text{Maximize } Z = 15X_1 + 6X_2 + 9X_3 + 2X_4$$

$$\text{Subject to, } 2X_1 + X_2 + 5X_3 + 6X_4 \le 20$$

$$3X_1 + X_2 + 3X_3 + 25X_4 \le 24$$

$$7X_1 + X_4 \le 70$$

$$X_1, X_2, X_3, X_4 \ge 0$$

Solution: Rewrite the inequality of the constraint into an equation by adding slack variables S_1, S_2, and S_3. The standard form of LPP becomes,

$$\text{Max } Z = 15X_1 + 6X_2 + 9X_3 + 2X_4 + 0S_1 + 0S_2 + 0S_3$$

$$\text{Subject to, } 2X_1 + X_2 + 5X_3 + 6X_4 + S_1 = 20$$

$$3X_1 + X_2 + 3X_3 + 25X_4 + S_2 = 24$$

$$7X_1 + X_4 + S_3 = 70$$

$$X_1, X_2, X_3, X_4, S_1, S_2, S_3 \ge 0$$

The initial basic feasible solution is, $S_1 = 20, S_2 = 24, S_3 = 70$

$(X_1 = X_2 = X_3 = X_4 = 0$ non-basic)

The intial simplex table is given by,

		C_j	15	6	9	2	0	0	0	
C_B	B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3	Min $\frac{X_B}{X_1}$
0	S_1	20	2	1	5	6	1	0	0	20/2=10
←0	S_2	24	(3)	1	3	25	0	1	0	24/3=8
0	S_3	70	7	0	0	1	0	0	1	70/7=10
	Z_j	0	0	0	0	0	0	0	0	
	$Z_j - C_j$		–15	–6	–9	–2	0	0	0	

$\therefore$ As some of $Z_j - C_j \le 0$, the current basic feasible solution is not optimum. $Z_1 - C_1 = -15$ is the most negative value, and hence, X_1 enters the basis and the variable S_2 leaves the basis.

First Iteration

		C_j	15	6	9	2	0	0	0	
C_B	B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3	Min $\frac{X_B}{X_2}$
←0	S_1	4	0	1/3	3	–32/3	1	–2/3	0	4/1/3=12
15	X_1	8	1	1/3	1	25/3	0	1/3	0	8/1/3=24
0	S_3	14	0	–7/3	–7	–172/3	0	–7/3	1	–
	Z_j	120	15	5	15	125	0	5	0	
	$Z_j - C_j$		0	–1↑	6	123	0	5	0	

NOTES

Since $Z_2 - C_2 = -1 < 0$; the solution is not optimal, and therefore, X_2 enters the basis and the basic variable S_1 leaves the basis.

Second Iteration

		C_j	15	6	9	2	0	0	0	
C_B	B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3	Min $\frac{X_B}{X_2}$
←0	S_1	4	0	1/3	3	–32/3	1	–2/3	0	4/1/3=12
15	X_1	8	1	1/3	1	25/3	0	1/3	0	8/1/3=24
0	S_3	14	0	–7/3	–7	–172/3	0	–7/3	1	–
	Z_j	120	15	5	15	125	0	5	0	
	$Z_j - C_j$		0	–1↑	6	123	0	5	0	

Since all $Z_j - C_j \geq 0$, the solution is optimal and is given by,

Max $Z = 132$, $X_1 = 4$, $X_2 = 12$, $X_3 = 0$, $X_4 = 0$

Example 5.23: Solve the following LPP using simplex method.

Maximize $Z = 3X_1 + 2X_2 + 5X_3$

Subject to, $X_1 + 2X_2 + X_3 \leq 430$

$3X_1 + 2X_3 \leq 460$

$X_1 + 4X_2 \leq 420$

$X_1, X_2, X_3 \geq 0$

Solution: Rewrite the constraint into an equation by adding slack variables S_1, S_2, S_3. The standard form of LPP becomes,

Maximize $Z = 3X_1 + 2X_2 + 5X_3 + 0S_1 + 0S_2 + 0S_3$

Subject to, $X_1 + 2X_2 + X_3 + S_1 = 430$

$3X_1 + 2X_3 + S_2 = 460$

$X_1 + 4X_2 + S_3 = 420$

$X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$

The initial basic feasible solution is,

$S_1 = 430$, $S_2 = 460$, $S_3 = 420$ ($X_1 = X_2 = X_3 = 0$)

Initial Table

		C_j	3	2	5	0	0	0	
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Min $\frac{X_B}{X_3}$
0	S_1	430	1	2	1	1	0	0	430/1=430
←0	S_2	460	3	0	(2)	0	1	0	460/2=230
0	S_3	420	1	4	0	0	0	1	
	Z_i	0	0	0	0	0	0	0	
	$Z_j - C_j$		–3	–2	–5↑	0	0	0	

NOTES

Since some of $Z_j - C_j \le 0$, the current basic feasible solution is not optimum. Since $Z_3 - C_3 = -5$ is the most negative, the variable X_3 enters the basis. To find the variable leaving the basis, find

$$\text{Min}\left(\frac{X_B}{X_{i3}}, X_{i3} > 0\right) = \text{Min}\left(\frac{430}{1} = 430, \frac{460}{2} = 230\right)$$

$\therefore$ The variable S_2 leaves the basis.

First Iteration

	C_j		3	2	5	0	0	0	
C_B	B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Min $\frac{X_B}{X_2}$
←0	S_1	200	–1/2	(2)	0	1	1/2	0	200/2=100
5	X_3	230	3/2	0	1	0	1/2	0	
0	S_3	420	1	4	0	0	0	1	420/4=105
	Z_i	1150	15/2	0	0	0	5/2	0	
	$Z_j - C_j$		9/2	–2↑	0	0	5/2	0	

Since $Z_2 - C_2 = -2$ is negative, the current basic feasible solution is not optimum. Therefore, the variable X_2 enters the basis and the variable S_1 leaves the basis.

M Method

In simplex algorithm, the *M* Method is used to deal with the situation where an infeasible starting basic solution is given. The simplex method starts from one Basic Feasible Solution (BFS) or the intense point of the feasible region of a Linear Programming Problem (LPP) presented in tableau form and extends to another BFS for constantly raising or reducing the value of the objective task till optimality is reached. Sometimes the starting basic solution may be infeasible, then *M* method is used to find the starting basic feasible solution (refer Example 5.23) each time it exists.

Example 5.24: Find a starting basic feasible solution each time it exists for the following LPP where there is no starting identity matrix using M method.

Maximize, $X_0 = C^T X$

Subject to, $AX = b$, $X \ge 0$; Where $b > 0$.

Solution: To get a starting identity matrix, we add artificial variables $X_{a1}, X_{a2}, \ldots\ldots, X_{am}$. The consequent values for the artificial variables can be *M* for maximization problem (where *M* is adequately large). This constant *M* will check artificial variables that will arise with positive values in the final optimal solutions. Now the LPP becomes,

Max $Z = C^T X - M \,.\, 1^T X_a$

Subject to, $AX + I_m X_a = b$,

$X \ge 0$

Where $X_a = (X_{a1}, X_{a2}, \ldots\ldots, X_{am})_T$ and 1 is the vector of all ones. Here, $X = 0$ and $X_a = b$ is the feasible starting basic feasible solution. For solving $AX + I_m X_a = b$, which is a solution to $AX = b$ we have to drive and take $X_a = 0$.

NOTES

Example 5.25: Using the linear programming given in the above example, solve the following LPP:

$$\text{Maximize,}\quad X_0 = X_1 + X_2$$
$$\text{Subject to,}\quad 2X_1 + X_2 \geq 4$$
$$X_1 + 2X_2 = 6$$
$$X_1, X_2 \geq 0$$

Solution: Add surplus variable X_3 and artificial variables X_4 and X_5, and then rewrite the equation as given below:

$$2X_1 + X_2 - X_3 + X_4 = 4$$
$$X_1 + 2X_2 + X_5 = 6$$
$$X_0 - X_1 - X_2 + M X_4 + M X_5 = 0$$

The columns corresponding to X_4 and X_5 form an identity matrix. This can be represented in tableau form as,

	X_1	X_2	X_3	X_4	X_5	***b***
X_4	2	1	–1	1	0	4
X_5	1	2	0	0	1	6
X_0	–1	–1	0	M	M	0

In the above table the row X_0 has the reduced cost coefficient for basic variables X_4 and X_5 which are not zero. First eliminate these nonzero entries to have the initial tableau.

	X_1	X_2	X_3	X_4	X_5	***b***
X_4	2	1	–1	1	0	4
X_5	1	2	0	0	1	6
X_0	$-(1 + 3M)$	$-(1 + 3M)$	M	0	0	$-10\,M$

The artificial variable becomes non-basic and can be dropped in subsequent calculations. Now the tableau becomes:

	X_1	X_2	X_3	X_5	***b***
X_1	1	1/2	–1/2	0	2
X_5	0	3/2	1/2	1	4
X_0	0	$-(1 + 3M)/2$	$-(1 + M)/2$	0	$2 - 4\,M$

Eliminating artificial variables we get,

	X_1	X_2	X_3	***b***
X_1	1	0	–2/3	2/3
X_2	0	1	1/3	8/3
X_0	0	0	–1/3	10/3

Now all the artificial variables are eliminated and $X = [2/3, 8/3, 0]^T$ is an initial basic feasible solution. Iterating again we get we following final optimal tableau:

	X_1	X_2	X_3	***b***
X_1	1	2	0	6
X_3	0	3	1	8
X_0	0	1	0	6

Hence, the optimal solution is $X = (6, 0, 8)^T$ with $X_0 = 6$.

5.7 SUMMARY

- Decision-making has always been very important in the business and industrial world, particularly with regard to the problems concerning production of commodities.
- English economist Alfred Marshall pointed out that the businessman always studies his production function and his input prices and substitutes one input for another till his costs become the minimum possible.
- Linear Programming (LP) is a major innovation since World War II in the field of business decision-making, particularly under conditions of certainty.
- The word 'Linear' means that the relationships are represented by straight lines, i.e., the relationships are of the form $y = a + bx$ and the word 'Programming' means taking decisions systematically.
- LP is a decision-making technique under given constraints on the assumption that the relationships amongst the variables representing different phenomena happen to be linear.
- The problem for which LP provides a solution may be stated to maximize or minimize for some dependent variable which is a function of several independent variables when the independent variables are subject to various restrictions.
- The applications of LPare numerous and are increasing every day. LPis extensively used in solving resource allocation problems. Production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning, etc., are some of its most fertile application areas.
- The term linearity implies straight line or proportional relationships among the relevant variables. Linearity in economic theory is known as constant returns which mean that if the amount of input doubles, the corresponding output and profit are also doubled.
- Process means the combination of particular inputs to produce a particular output. In a process, factors of production are used in fixed ratios, of course, depending upon technology and as such no substitution is possible with a process.
- Criterion function is also known as objective function which states the determinants of the quantity either to be maximized or to be minimized.
- LP model is based on the assumptions of proportionality, additivity, certainty, continuity and finite choices.
- The applications of linear programming problems are based on linear programming matrix coefficients and data transmission prior to solving the simplex algorithm.
- The problem can be formulated from the problem statement using linear programming techniques.

NOTES

Check Your Progress

9. When is an objective function minimized? When is it maximized?
10. What is meant by a feasible solution?
11. What is a feasible region?
12. What is an optimal solution?
13. What are non-degenerate and degenerate type basic feasible solutions?
14. Define the simplex method.
15. How is a leaving element converted to unity in a simplex algorithm?
16. What is the role of the slack variable?
17. When M method is used?

NOTES

- Linear programming problems are associated with the efficient use of allocation of limited resources to meet desired objectives. A solution required to solve the linear programming problem is termed as optimal solution.
- Linear programming problem may be solved using a simplified version of the simplex technique called transportation method. Because of its major application in solving problems involving several product sources and several destinations of products, this type of problem is frequently called the transportation problem.
- The goal of the diet problem is to find the cheapest combination of foods that will satisfy all the daily nutritional requirements of a person.
- The problem is formulated as a linear program where the objective is to minimize cost and meet constraints which require that nutritional needs be satisfied.
- The portfolio optimization template calculates the optimal capital of investments that gives the highest return for the least risk. The unique design of the portfolio optimization technique helps in financial investments or business portfolios.
- Crew scheduling is an important application of linear programming problem. It helps if any airline has a problem related to a large potential crew schedules variables.
- The general LPP can be put in either canonical or standard forms.
- In the standard form, irrespective of the objective function, namely maximize or minimize, all the constraints are expressed as equations. Moreover, RHS of each constraint and all variables are non-negative.
- In the canonical form, if the objective function is of maximization, all the constraints other than non-negativity conditions are '$\leq$' type. If the objective function is of minimization, all the constraints other than non-negative conditions are '$\geq$' type.
- Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be that at the previous vertex.
- In simplex algorithm, the *M* Method is used to deal with the situation where an infeasible starting basic solution is given.
- The simplex method starts from one Basic Feasible Solution (BFS) or the intense point of the feasible region of a Linear Programming Problem (LPP) presented in tableau form and extends to another BFS for constantly raising or reducing the value of the objective task till optimality is reached.

5.8 KEY TERMS

- **Linear programming:** A decision-making technique under a set of given constraints and is based on the assumption that the relationships amongst the variables representing different phenomena are linear
- **Decision variables:** Variables that form objective function and on which the cost or profit depends
- **Linearity:** Straight line or proportional relationships among the relevant variables. Linearity in economic theory is known as constant return
- **Process:** The combination of one or more inputs to produce a particular output

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- **Criterion function:** An objective function which states the determinants of the quantity to be either maximized or minimized
- **Constraints:** Limitations under which planning is decided. Restrictions imposed on decision variables
- **Feasible solution:** Any solution to a LPP which satisfies the non-negativity restrictions of the LPP
- **Feasible region:** The region comprising all feasible solutions
- **Optimal solution:** Any feasible solution which optimizes (minimizes or maximizes) the objective function of the LPP
- **Proportionality:** An assumption made in the objective function and constraint inequalities. In economic terminology this means that there are constant returns to scale
- **Certainty:** Assumption that includes prior knowledge of all the coefficients in the objective function, the coefficients of the constraints and the resource values. LP model operates only under conditions of certainty.
- **Additivity:** An assumption which means that the total of all the activities is given by the sum total of each activity conducted separately
- **Continuity:** An assumption which means that the decision variables are continuous
- **Finite choices:** An assumption that implies that finite numbers of choices are available to a decision-maker and the decision variables do not assume negative values
- **Solution:** A set of values $X_1, X_2, ..., X_n$ which satisfies the constraints of the LPP
- **Basic solution:** In a given system of m linear equations with n variables ($m < n$), any solution which is obtained by solving m variables keeping the remaining $n - m$ variables zero is called a basic solution
- **Basic feasible solution:** A basic solution which also satisfies the condition in which all basic variables are non-negative
- **Canonical form:** It is irrespective of the objective function. All the constraints are expressed as equations and right hand side of each constraint and all variables are non-negative
- **Slack variables:** If the constraints of a general LPP be given as $\Sigma a_{ij} X_i \leq b_i$ ($i = 1, 2, ..., m$; $j = 1, 2, ..., n$), then the non-negative variables S_i is introduced to convert the inequalities '≤' to the equalities are called slack variables
- **Surplus variables:** If the constraints of a general LPP be $\Sigma a_{ij} X_i \geq b_i$ ($i = 1, 2, ..., m$; $j = 1, 2, ..., n$), then non-negative variables S_i introduced to convert the inequalities '≥' to the equalities are called surplus variables

5.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Linear programming is a decision-making technique under a set of given constraints and is based on the assumption that the relationships amongst the variables representing different phenomena are linear.
2. Criterion function is objective function which states the determinants of the quantity, to be either maximized or minimized.

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3. Linear programming finds application in agricultural and various industrial problems.
4. Constraints are limitations under which planning is decided, these are restrictions imposed on decision variables.
5. Solution of a linear programming is a set of values $X_1, X_2, ..., X_n$, satisfying the constraints of the LPP is called its solution.
6. In a given a system of m linear equations with n variables ($m < n$), any solution which is obtained by solving m variables keeping the remaining $n - m$ variables zero is called a basic solution.
7. In a given a system of m linear equations with n variables ($m < n$), where m variables are solved, keeping remaining $n - m$ variables zero, m variables are called basic variables and the remaining variables are called non-basic variables.
8. Basic feasible solution is a basic solution which also satisfies the condition in which all basic variables are non-negative.
9. An objective function is maximized when it is a profit function. It is minimized when it is a cost function.
10. Feasible solution of a LPP is a solution that satisfies the non-negativity restrictions of the LPP.
11. Feasible region is the region comprising all feasible solutions.
12. Optimal solution of a LPP is a feasible solution which optimizes (minimizes or maximizes) the objective function of the LPP.
13. Non-degenerate and degenerate solutions are basic feasible solutions. In a problem which has exactly m positive variables, X_i ($i = 1, 2, ..., m$), i.e., none of the basic variables is zero, then it is called non-degenerate type and if one or more basic variables are zero, such basic feasible solution is said to be degenerate type.
14. Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be that at the previous vertex.
15. The leaving element is converted to unity by dividing the key equation by the key element and all other elements in its column to zero by using the formula:

    New element

    $$= \text{Old element} - \left[\frac{\text{Product of elements in key row and key column}}{\text{Key element}}\right]$$

16. By introducing slack variable, the problem is converted into standard form.
17. M method is used to find the starting basic feasible solution each time it exists when an infeasible starting basic solution is given.

5.10 QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is meant by proportionality in linear programming?
2. What do you understand by certainty in linear programming?

3. What is meant by continuity in linear programming?
4. What are finite choices in the context of linear programming?
5. What are the basic constituents of an LP model?
6. What is the canonical form of a LPP?
7. What are characteristics of the canonical form?
8. What are slack variables? Where are they used? Explain in brief.
9. What do you understand by surplus variables?
10. What is the simplex method?
11. Does every LPP solution have an optimal solution? Explain.
12. What is the importance of the *M* method?

Long-Answer Questions

1. A company manufactures 3 products *A*, *B* and C. The profits are: ` 3, ` 2 and ` 4 respectively. The company has two machines and given below is the required processing time in minutes for each machine on each product.

	Products		
Machines	***A***	***B***	***C***
I	4	3	5
II	2	2	4

Machines I and II have 2000 and 2500 minutes respectively. The company must manufacturers 100 *A*'s 200 *B*'s and 50 *C*'s but no more than 150 *A*'s. Find the number of units of each product to be manufactured by the company to maximize the profit. Formulate the above as a LP Model.

2. A company produces two types of leather belts *A* and *B*. *A* is of superior quality and *B* is of inferior quality. The respective profits are ` 10 and ` 5 per belt. The supply of raw material is sufficient for making 850 belts per day. For belt *A*, a special type of buckle is required and 500 are available per day. There are 700 buckles available for belt *B* per day. Belt *A* needs twice as much time as that required for belt *B* and the company can produce 500 belts if all of them were of the type *A*. Formulate a LP Model for the given problem.
3. The standard weight of a special purpose brick is 5 kg and it contains two ingredients B_1 and B_2, where B_1 costs ` 5 per kg and $B_2$ costs ` 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg of B_1 and a minimum of 2 kg of B_2 since the demand for the product is likely to be related to the price of the brick. Formulate the given problem as a LP Model.
4. Egg contains 6 units of vitamin *A* per gram and 7 units of vitamin *B* per gram and 12 units of vitamin *B* per gram and costs 20 paise per gram. The daily minimum requirement of vitamin *A* and vitamin *B* are 100 units and 120 units respectively. Find the optimal product mix.
5. In a chemical industry two products *A* and *B* are made involving two operations. The production of *B* also results in a by product *C*. The product *A* can be sold at ` 3 profit per unit and *B* at ` 8 profit per unit. The by product *C* has a profit of ` 2 per unit but it cannot be sold as the destruction cost is Re 1 per unit. Forecasts show that upto 5 units of *C* can be sold. The company gets 3 units of *C* for each units of *A* and *B* produced. Forecasts show that they can sell all the units of *A* and

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B produced. The manufacturing times are 3 hours per unit for A on operation one and two respectively and 4 hours and 5 hours per unit for B on operation one and two respectively. Because the product C results from producing B, no time is used in producing C. The available times are 18, and 21 hours of operation one and two respectively. How much of A and B need to be produced keeping C in mind, to make the highest profit. Formulate the given problem as LP Model.

6. A company produces two types of hats. Each hat of the first type requires as much labour time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are ` 8 for type B, formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.

7. A company desires to devote the excess capacity of the three machines lathe, shaping machine and milling machine to make three products A, B and C. The available time per month in these machinery are tabulated below:

Machine	***Lathe***	***Shaping***	***Milling***
Available Time/Month	200 hrs	l00 hrs	180 hrs

The time taken to produce each unit of the products A, B and C on the machines is displayed in the table below.

	Lathe	***Shaping***	***Milling***
Product A hrs	6	2	4
Product B hrs	2	2	–
Product C hrs	3	_	3

The profit per product would be ` 20, ` 16 and ` 12 respectively on product A, B and C.

Formulate a LPP to find the optimum product-mix.

8. An animal food company must produce 200 kg of a mixture consisting of ingredients X_1 and X_2 daily. X_1 costs ` 3/- per kg and $X_2$ ` 8/- per kg. No more than 80 kg of X_1 can be used and at least 60 kg of X_2 must be used. Formulate a LP model to minimize the cost.

9. Solve the following by graphical method:

(i) Max $Z = X_1 - 3X_2$

Subject to,
$$X_1 + X_2 \le 300$$
$$X_1 - 2X_2 \le 200$$
$$2X_1 + X_2 \le 100$$
$$X_2 \le 200$$
$$X_1, X_2 \ge 0$$

(ii) Max $Z = 5X + 8Y$

Subject to,
$$3X + 2Y \le 36$$
$$X + 2Y \le 20$$
$$3X + 4Y \le 42$$
$$X, Y \ge 0$$

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(iii) Max $Z = X - 3Y$

Subject to, $X + Y \leq 300$

$X - 2Y \leq 200$

$X + Y \leq 100$

$Y \geq 200$

and $X, Y \geq 0$

10. Solve graphically the following LPP:

Max $Z = 20X_1 + 10X_2$

Subject to, $X_1 + 2X_2 \leq 40$

$3X_1 + X_2 \geq 30$

$4X_1 + 3X_2 \geq 60$

and $X_1, X_2 \geq 0$

11. A company produces two different products *A* and *B*. The company makes a profit of ` 40 and ` 30 per unit on *A* and *B* respectively. The production process has a capacity of 30,000 man hours. It takes 3 hours to produce one unit of *A* and one hour to produce one unit of *B*. The market survey indicates that the maximum number of units of product *A* that can be sold is 8000 and those of *B* is 12000 units. Formulate the problem and solve it by graphical method to get maximum profit.

12. Solve graphically the following LPP:

Min $Z = 3X - 2Y$

Subject to, $-2X + 3Y \leq 9$

$X - 5Y \geq -20$

$X, Y \geq 0$

(i) Min $Z = -6X_1 - 4X_2$

Subject to, $2X_1 + 3X_2 \geq 30$

$3X_1 + 2X_2 \leq 24$

$X_1 + X_2 \geq 3$

$X_1, X_2 \geq 0$

(ii) Max $Z = 3X_1 - 2X_2$

Subject to, $X_1 + X_2 \leq 1$

$2X_1 + 2X_2 \geq 4$

$X_1, X_2 \geq 0$

(iii) Max $Z = -X_1 + X_2$

Subject to, $X_1 - X_2 \geq 0$

$-3X_1 + X_2 \geq 3$

$X_1, X_2 \geq 0$

13. Using simplex method, find non-negative values of X_1, X_2 and X_3 when

(i) Max $Z = X_1 + 4X_2 + 5X_3$

Subject to the constraints,

$3X_1 + 6X_2 + 3X_3 \leq 22$

$X_1 + 2X_2 + 3X_3 \leq 14$ and

$3X_1 + 2X_2 \leq 14$

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(ii) Max $Z = X_1 + X_2 + 3X_3$

Subject to, $3X_1 + 2X_2 + X_3 \leq 2$

$2X_1 + X_2 + 2X_3 \leq 2$

$X_1, X_2, X_3 \geq 0$

(iii) Max $Z = 10X_1 + 6X_2$

Subject to, $X_1 + X_2 \leq 2$

$2X_1 + X_2 \leq 4$

$3X_1 + 8X_2 \leq 12$

$X_1, X_2 \geq 0$

(*iv*) Max $Z = 30X_1 + 23X_2 + 29X_3$

Subject to the constraints,

$6X_1 + 5X_2 + 3X_3 \leq 52$

$6X_1 + 2X_2 + 5X_3 \leq 14$

$X_1, X_2, X_3 \geq 0$

(*v*) Max $Z = X_1 + 2X_2 + X_3$

Subject to, $2X_1 + X_2 - X_3 \geq -2$

$-2X_1 + X_2 - 5X_3 \leq 6$

$4X_1 + X_2 + X_3 \leq 6$

$X_1, X_2, X_3 \geq 0$

14. A manufacturer is engaged in producing 2 products *X* and *Y*, the contribution margin being ` 15 and ` 45 respectively. *A* unit of product *X* requires 1 unit of facility *A* and 0.5 unit of facility *B*. *A* unit of product *Y* requires 1.6 units of facility *A*, 2.0 units of facility *B* and 1 unit of raw material *C*. The availability of total facility *A*, *B* and raw material *C* during a particular time period are 240, 162 and 50 units respectively.

    Find out the product-mix which will maximize the contribution margin by simplex method.

15. A firm has available 240, 370 and 180 kg of wood, plastic and steel respectively. The firm produces two products *A* and *B*. Each unit of *A* requires 1, 3 and 2 kg of wood, plastic and steel respectively. The corresponding requirement for each unit of *B* are 3, 4 and 1 respectively. If *A* sells for ` 4, and *B* for ` 6, determine how many units of *A* and *B* should be produced in order to obtain the maximum gross income. Use the simplex method.

16. Solve the following LPP applying *M* method:

    Maximize $Z = 3X_1 + 4X_2$

    Subject to, $2X_1 + X_2 \leq 600$

    $X_1 + X_2 \leq 225$

    $5X_1 + 4X_2 \leq 1000$

    $X_1 + 2X_2 \geq 150$

    $X_1, X_2 \geq 0$

5.11 FURTHER READING

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